MATH655 EXERCISES: FORCING RESPECTS LOGICAL EQUIVALENCE

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The purpose of this set of exercises is to see that forcing respects logical equivalence. This will let us conclude that if φ and ψ are logically equivalent then whenever $p \Vdash \varphi$ we may freely assume $p \Vdash \psi$. This is, to put it lightly, useful to be able to do. It also ensures that when defining the forcing relation it did not matter which choice of basic logical connectives we used. So the fact that we used a minimal set of connectives/quantifiers—in our case, $\neg, \wedge$, and $\forall$—is not affecting the theory.

First it must be made clear what it means for two formulae to be logically equivalent.

Definition 1. Let $\varphi(x, y, \ldots)$ and $\psi(x, y, \ldots)$ be two formulae in the same language. Then they are *logically equivalent* if given any structure M in this language and any selection of elements $a, b, \ldots$ from M we have $M \models \varphi(a, b, \ldots)$ iff $M \models \psi(a, b, \ldots)$.

We could equivalently defined that φ and ψ are logically equivalent if

$$\emptyset \models \forall x \forall y \ldots \ \varphi(x, y, \ldots) \Leftrightarrow \ \psi(x, y, \ldots),$$

where $\emptyset$ is the empty theory with no axioms.

The first exercise is to check that logical equivalence is a strong condition.

Exercise 2. Show that the following pairs of formulae are not logically equivalent.

(1) $x \neq y$ and $x < y \vee y < x$;
(2) $x = y$ and $\forall z \ z \in x \Leftrightarrow z \in y$;
(3) $\forall y \ x + y = y$ and $x + x = x$.
(4) $(x + y) + z = w$ and $x + (y + z) = w$.

Probably, you solved the previous exercies by (spoilers!) finding structures which witness the non-equivalence of the pairs of formulae. On the other hand, showing that two formulae are logically equivalent would require looking at all (countable) structures, which is hard in general. So we want a nicer criterion for logical equivalence.

It follows from the completeness and soundness theorems for first-order logic that φ and ψ are logically equivalent iff

$$\emptyset \vdash \forall x \forall y \ldots \ \varphi(x, y, \ldots) \Leftrightarrow \ \psi(x, y, \ldots).$$

So if we explicate a proof system we will have a method for checking logical equivalence which doesn't require looking at lots and lots of structures. Let's do so now.

There's two main approaches one can take with developing a proof system. The first is to try to make it as close to normal human reasoning as possible. The second is to not do that. Some advantages of the first approach are that using it to prove things is easier, and it's clearer why it formally captures the intuitive notion of proof. One disadvantage is that such proof systems are more complex, since as humans we use lots and lots of rules and proof techniques. This makes these

Date: March 4, 2019.

systems less amenable to metalogical analysis, because there's more cases to consider and more to prove.

We will take the latter approach here. We will moreover take the approach of breaking up our proof system into two parts: logical axioms and rules of inference. The next exercise illustrates the issue with deciding what to pick as logical axioms. A clear desideratum is that a logical axiom should be *logically valid*—true in every structure. Is that enough?

Exercise 3 (Only do if you have some background in computability theory). Prove that there is no Turing machine that decides which formulae, in a fixed finite language, are logically valid. (This problem is known as Hilberts *Entscheidungsproblem*, and was famously shown by Church and Turing to lack a computable answer.)

So what Arnold Miller called the "Mickey Mouse" proof system of taking all logical validities as logical axioms will not do. Instead we have to restrict which logical validities we take as axioms. A good exercise is to try on your own to come up with a computable set of logical validities from which you can derive all others. But I will just give one possible answer. I will specialize to the case of the language of set theory, allowing the possibility of constants. (We want to allow this possibility for when we introduce $\mathbb{P}$-names.)

We will take the universal closure of the following as logical axioms, where the universal closure of φ is obtained by putting universal quantifiers in front for every free variable.

(1) Equality axioms. Here r, s, t are terms. (In case the language only has relation symbols, e.g. the language of set theory, terms are just variables. If the language includes function and constant symbols, then terms are built up from applying functions to variables and constants.)
   (a) $t = t$.
   (b) $s = t \Rightarrow t = s$.
   (c) $s = t \Rightarrow (r = s \Rightarrow r = t)$.
   (d) $s = t \Rightarrow (r \in s \Leftrightarrow r \in t)$.
(2) Quantifier axioms. Here t is a term and x is a variable.
   (a) $\forall x \varphi(x) \Rightarrow \varphi(t)$.
   (b) $\varphi \Rightarrow \forall x \varphi(x)$ when x is not a variable in φ.
(3) Material conditional axioms. Here x is a variable.
   (a) $\forall x(\varphi \Rightarrow \psi) \Rightarrow (\forall x \varphi \Rightarrow \forall x \psi)$.
   (b) $\varphi \Rightarrow (\psi \Rightarrow \varphi)$.
   (c) $(\neg\psi \Rightarrow \neg\varphi) \Rightarrow (\varphi \Rightarrow \psi)$.
   (d) $[\varphi \Rightarrow (\psi \Rightarrow \theta)] \Rightarrow [(\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \theta)]$.

It is manifest that the set of these logical axioms is computable, since I listed out schemata they must belong to.

Exercise 4. Write, in ordinary English, the proof techniques/rules of inference associate with each of these logical axioms.

We will take a single rule of inference, namely *modus ponens*.

(1) From $T \vdash \varphi$ and $T \vdash \varphi \Rightarrow \psi$ conclude $T \vdash \psi$.

We can use these logical axioms and rule of inference to define $\vdash$.

Definition 5. The relation $T \vdash \varphi$ is the smallest relation satisfying the following properties.

- If φ is a logical axiom then $T \vdash \varphi$.

- If $\varphi \in T$ then $T \vdash \varphi$. (These are the non-logical axioms.)
- If $T \vdash \varphi$ and $T \vdash \varphi \Rightarrow \psi$ then $T \vdash \psi$.

Equivalently, $T \vdash \varphi$ if there is a finite sequence of formuale

$$\sigma_0, \ldots, \sigma_k$$

so that σ_k is φ and each σ_i is either a logical axiom, in T, or else there are $j, \ell < i$ so that σ_ℓ is $\sigma_j \Rightarrow \sigma i$.

Exercise 6. Prove the deduction theorem: $T \vdash \varphi \Rightarrow \psi$ iff $T \cup \{\varphi\} \vdash \psi$.

Exercise 7. Prove Gödel's completeness theorem that every consistent theory has a model. (Hint: first expand the language of your consistent theory T to have countably many new constant symbols c_n. Then show that T can be extended to a consistent theory S with new axioms $\exists x \varphi(x) \Rightarrow \varphi(c_n)$ for each φ in the language with the extra constant symbols. Next show that S can be extended to a theory $\bar{S}$ which is $\subseteq$-maximal among the consistent theories. (Hint: use weak Kőnig's lemma.) Finally, define an equivalence relation on $\{c_n\}$ as $c_n \sim c_m$ if "$c_n = c_m$" $\in \bar{S}$. Use the $\sim$-equivalence classes as the domain for a model of T.)

Having clarified just what we mean by logical equivalence, let us now turn to forcing. We work here with conditions $p \in \mathbb{P}$ with $\mathbb{P}$-names $\sigma, \tau, \ldots$. This is the most important exercise of this set.

Exercise 8. Show that if φ and ψ are logically equivalent then $p \Vdash \varphi$ iff $p \Vdash \psi$. (Hint: due to symmetry, you only have to prove one direction of this. So you want to show that if $\varphi \Rightarrow \psi$ is logically valid then $p \Vdash \varphi$ implies $p \Vdash \psi$. You do this by showing that $\Vdash$ respects *modus ponens*. That is, if $p \Vdash \varphi$ and $p \Vdash \varphi \Rightarrow \psi$ then $p \Vdash \psi$. And you want to show every condition forces every logical axiom. Many of these are one line arguments. For example, if $p \Vdash \forall x \varphi(x)$ then $p \Vdash \varphi(\tau)$ for any $\mathbb{P}$-name τ, just by the definition of the forcing relation for universal quantifiers.)

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