# SOLUTION FOR PROBLEM 3 IN MATH 454 TAKE-HOME FINAL

**Problem 3.** *Let* $W = \{R \subseteq \omega \times \omega : (\operatorname{dom} R, R)$ *is a well-order*$\}$. *Let* $\preceq$ *be the binary relation on* $W$ *defined as* $R \preceq S$ *if there is an order-embedding* $f : (\operatorname{dom} R, R) \to (\operatorname{dom} S, S)$. *Consider* $W/{\cong}$, *where you quotient out* $W$ *by identifying isomorphic well-orders.* $(a)$ *Show that* $\preceq$ *is a congruence modulo* $\cong$; *that is, show that if* $(\operatorname{dom} R, R) \cong (\operatorname{dom} R', R')$, *and* $(\operatorname{dom} S, S) \cong (\operatorname{dom} S', S')$ *and* $R \preceq S$, *then* $R' \preceq S'$. $(b)$ *Show that* $(W/{\cong}, \prec)$ *is a well-order and determine the ordertype of* $(W/{\cong}, \prec)$, *where* $R \prec S$ *if* $R \preceq S$ *but* $R \not\cong S$.

*Solution.* $(a)$ Let $f : (\operatorname{dom} R, R) \to (\operatorname{dom} R', R')$ and $g : (\operatorname{dom} S, S) \to (\operatorname{dom} S', S')$ be isomorphisms and let $e : (\operatorname{dom} R, R) \to (\operatorname{dom} S, S)$ be an order-embedding. Then $g \circ e \circ f^{-1} : (\operatorname{dom} R', R') \to (\operatorname{dom} S', S')$ is an embedding.

$(a)$ (Alternate Proof) Because $R$ and $R'$ are isomorphic, they are both isomorphic to the same ordinal, call it $\rho$. And similarly $S$ and $S'$ are both isomorphic to some ordinal $\sigma$. Because $R$ order-embeds into $S$ we get that $\rho \leq \sigma$. But because $\rho \leq \sigma$ we get that $R'$ order-embeds into $S'$.

$(b)$ First let us see that $(W/{\cong}, \prec)$ is a well-order. That $\prec$ is irreflexive is immediate from the definition. That it is transitive is clear from composing embeddings. To see that it is total, take $R, S \in W$. Let $\rho$ be the ordinal isomorphic to $R$ and $\sigma$ be the ordinal isomorphic to $S$. Then either $\rho < \sigma$, $\rho = \sigma$, or $\rho > \sigma$. These three possibilities correspond to $R \prec S$, $R \cong S$, or $R \succ S$, showing that $\prec$ is total on the equivalence classes. Finally, we want to see that $\prec$ is well-founded. Suppose toward a contradiction we have a descending sequence

$$R_0 \succ R_1 \succ \cdots \succ R_n \succ \cdots$$

in $W$. Let $\rho_n$ be the ordinal isomorphic to $R_n$. Then we have

$$\rho_0 > \rho_1 > \cdots > \rho_n > \cdots,$$

an infinite descending sequence in the ordinals which is impossible.

Now let us see that the ordertype of $(W/{\cong}, \prec)$, call it $\gamma$ is $\omega_1$. First, let us see that if $\alpha$ is a countable ordinal then $\alpha \leq \gamma$. To see this, note that because $\alpha$ is countable there is $R \in W$ so that $R$ is isomorphic to $\alpha$, via an isomorphism $f : \alpha \to (\operatorname{dom} R, R)$. Now consider $\beta \leq \alpha$. Look at $S_\beta = R \upharpoonright f[\beta]$. Then $S_\beta$ is a well-order of a subset of $\omega$, i.e. $S_\beta \in W$. And if $\beta < \beta' \leq \alpha$ then $S_\beta \prec S_{\beta'}$. So we have an isomorphic copy of $\alpha$ inside $W$, so $\alpha \leq \gamma$.

Next, observe that $\gamma \leq \omega_1$, because every well-order in $W$ is countable. So together we get that $\gamma = \omega_1$. □

*Remark.* You can do a similar construction to get an explicit well-order of ordertype $\omega_2$, $\omega_3$, and so on. In general, if $X$ is any infinite, well-orderable set then you get a well-order with ordertype $|X|^+$ by considering the collection of well-orders of subsets of $X$ ordered by embeddability.