MATH454: HOMEWORK 0

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Exercise 1. *Show that there is no bijection between* $\mathbb{N}$ *and* $\mathbb{R}$.

Solution. It was proven in class on Tuesday, August 27 that $\mathbb{R}$ is in bijective correspondence with $\mathcal{P}(\mathbb{N})$. Accordingly, it suffices to prove that there is no bijection between $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$. To this end, consider an arbitrary function $f\colon \mathbb{N} \to \mathcal{P}(\mathbb{N})$. Define $D \subseteq \mathbb{N}$ as $D = \{n \in \mathbb{N} : n \notin f(n)\}$. I claim that D is not in the range of f. To see this, suppose it were the case that $D = f(n)$ for some n. If $n \in D$, then by the definition of D we would have that $n \notin f(n) = D$, a contradiction. On the other hand, if $n \notin D = f(n)$, then again by by the definition of D we would have that $n \in D$. Either way, we get a contradiction. So it must be that our arbitrary f is not surjective, and hence fails to be bijective. □

Exercise 2. *Show that if* $X \subseteq \mathbb{R}$ *then either* X *injects into* $\mathbb{N}$ *or else* $\mathbb{R}$ *injects into* X.

Solution. I was unable to find a proof of this assertion in full generality. However, I was able to prove it in the case where X contains a nondegenerate interval.

Suppose X contains a nondegenerate interval I. By shrinking I if necessary, we way assume without loss that I is an open interval, that is of the form (a, b). Observe that (a, b) is in bijective correspondence with with $(0, 1)$, simply by shifting and rescaling. Thus, (a, b) is in bijective correspondence with any other interval. Next, note that restricting the tangent function gives a bijection from $(-\pi/2, \pi/2)$ to $\mathbb{R}$. Composing these bijections we get that (a, b) is in bijective correspondence with $\mathbb{R}$. This then yields an injection of $\mathbb{R}$ into X. □

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