MATH454 HOMEWORK 7
DUE THURSDAY, OCTOBER 17

In class we saw that all countable dense linear orders without endpoints are isomorphic. In this homework you will show that this is not true for uncountable linear orders.

Let L_H be the linear order $\omega_1 \times [0, 1)$ using the lexicographic order.

Exercise 1. Show that L_H is a dense linear order with a left endpoint but no right endpoint.[1] And show that $|L_H| = 2^{\aleph_0}$.

Exercise 2. Show that L_H has the supremum property. That is, show that if $X \subseteq L_H$ has an upper bound in L_H then X has a supremum in L_H.

Let L be the dense lineor order without endpoints obtained from L_H by removing the left endpoint. (L is known as the *long line*. It is a nice source of counterexamples in topology.)

Exercise 3. Show that L and $\mathbb{R}$ are not isomorphic as orders. [Hint: in an earlier homework assignment you were asked to show that any well-ordered suborder of $\mathbb{R}$ must be countable. Show that this does not hold for L.]

This shows that dense linear orders without endpoints of cardinality $2^{\aleph_0}$ need not be isomorphic, even if they both have the supremum property. Next you will show that for any uncountable cardinal κ there are non-isomorphic dense linear orders (without endpoints) of cardinality κ. First, you need to prove a lemma. For a linear order A and $a \in A$, let $\mathrm{pred}_A(a) = \{x \in A : x < a\}$ be the set of predecessors of a.

Exercise 4. Let A and B be linear orders and suppose $f : A \to B$ is an isomorphism. Show that for any $a \in A$ that $f[\mathrm{pred}_A(a)] = \mathrm{pred}_B(f(a))$.

Exercise 5. Fix an uncountable cardinal κ. Show that $A = \kappa \times \mathbb{Q}$ and $B = (\kappa + \kappa) \times \mathbb{Q}$, where both have the lexicographic order, are not isomorphic. [Hint: First prove that if $a \in A$ then $\mathrm{pred}_A(a)$ has cardinality $< \kappa$. Is the same true for B?]

[1] The H in L_H is for "half-open".