MATH454 HOMEWORK 5
DUE THURSDAY, OCTOBER 3

Exercise 1. Do exercise 4.2 from the textbook.

Exercise 2. Do exercise 4.3 from the textbook.

Exercise 3. Do exercise 4.8 from the textbook.

Exercise 4. Do exercise 4.12 from the textbook.

The remaining exercises are about equivalent forms of the axiom of choice. For these exercises, your base theory should be ZF. That is, you are not allowed to use the axiom of choice unless it's the antecedent in a conditional.

In class we proved that the axiom of choice implies the well-ordering theorem.

Exercise 5. Show that the well-ordering theorem implies the axiom of choice. That is, assume that for every set X there is $< \subseteq X \times X$ which is a well-order on X. Show that the axiom of choice holds.

Exercise 6. Exercise 4.16 in the textbook is about Zorn's lemma, asking you to prove that the axiom of choice implies Zorn's lemma. Do this exercise. (See the textbook for definitions and a hint.)

Exercise 7. Prove that Zorn's lemma implies the axiom of choice. [Hint: given a set X, consider the partial order of all well-orders on a subset of X, ordered by extension. Show that chains in this partial order have upper bounds. Then apply Zorn's lemma, then Exercise 5.]

Cardinal trichotomy is the statement for any two sets X and Y either there is an injection $f : X \to Y$ or there is an injection $f : Y \to X$ (or possibly both). In class we saw that the axiom of choice implies cardinal trichotomy. The next two exercises ask you to prove the reverse implication. Remember that you are working in ZF!

Definition. *Given a set X, the* Hartogs number *of X, denoted $\aleph(X)$, is the smallest ordinal α so that there is no injection $f : \alpha \to X$.*

Exercise 8 (Reach). Show, working from ZF, that $\aleph(X)$ always exists for every set. [Hint: consider the set $\{\alpha \in \text{Ord} : \text{there is an injection } \alpha \to X\}$. Show that this set is $\aleph(X)$.]

Exercise 9 (Reach). Show that cardinal trichotomy implies the axiom of choice. [Hint: use the previous exercise.]

The following reach exercise is for those who have taken a class in abstract algebra.

Exercise 10 (Reach). Show that the axiom of choice is equivalent to the statement that for every nonempty set X there is a binary operation $\star : X \times X \to X$ so that $(X, \star)$ is a group. [Hint: for the forward implication, consider the group $G = \bigoplus_{x \in X} \mathbb{Z}$, the direct sum of a copy of $\mathbb{Z}$ for every element of X. Show that if X is infinite then there is a bijection between G and X. For the backward implication, consider a group structure on $\{0\} \times X \cup \{1\} \times \aleph(X)$. Use this to show that X injects into $\aleph(X)$.]