MATH454 HOMEWORK 4
DUE THURSDAY, SEPTEMBER 26

Exercise 1. Do Exercise 3.17 from the textbook.

Define a new operation on ordinals, *tetration* $\alpha \uparrow \beta$, by recursion:

- $\alpha \uparrow 0 = 1$;
- $\alpha \uparrow (\beta + 1) = \alpha^{\alpha \uparrow \beta}$;
- If γ is limit, then $\alpha \uparrow \gamma = \sup_{\beta < \gamma} \alpha \uparrow \beta$.

Exercise 2. Calculate $2 \uparrow 4$ and $\omega \uparrow \omega$.

Exercise 3. Show by example that $(\alpha \uparrow \beta) \uparrow \gamma = \alpha \uparrow (\beta \cdot \gamma)$ and $(\alpha \cdot \beta) \uparrow \gamma = (\alpha \uparrow \gamma) \cdot (\beta \uparrow \gamma)$ do not hold for all ordinals. [Hint: You can find finite counterexamples.]

Exercise 4. Prove the following facts about tetration:

- $1 \uparrow \alpha = 1$ for all α;
- If $\beta < \gamma$ then $\alpha \uparrow \beta \leq \alpha \uparrow \gamma$. [Hint: Fix α and β and do induction on $\gamma \geq \beta$. What is the base case?]

The remaining exercise about induction and recursion in a more general setting than linear orders.

You can do induction and recursion on more than linear orders. For linear orders, it was precisely the well-orders on which induction is valid. For relations in general, it is precisely the well-founded relations on which induction is valid. Recall that a binary relation $R \subseteq A \times A$ is *well-founded* if any nonempty $X \subseteq \operatorname{dom} R$ has an R-minimal element. That is, if $X \subseteq \operatorname{dom} R$ is nonempty then there is $m \in X$ so that there is no $x \in X$ with $x \mathrel{R} m$.

Exercise 5. Prove that induction is valid on well-founded relations. That is, let R be a well-founded relation with domain A. Suppose that $X \subseteq A$ has the property that for all $x \in A$ if all $y \mathrel{R} x$ are in X, then $x \in X$. Show that $X = A$. [Hint: Suppose toward a contradiction that $X \neq A$. Then $A \setminus X$ has a minimal element.]

Similar to how you can do induction on Ord, even though Ord is not a set, you can do induction on some well-founded relations too big to be sets. The most important example of this is induction on the membership relation $\in$.

Exercise 6. Prove that induction on $\in$ is valid. That is, let $P(x)$ be a property. Suppose that for all sets x that if all $y \in x$ have $P(y)$ then $P(x)$. Show that $P(x)$ holds for all x. [Hint: Suppose there is some x so that $P(x)$ fails. Then, since the membership relation is well-founded, there is a minimal counterexample: an x so that $P(x)$ fails but $P(y)$ holds for all $y \in x$.]

Exercise 7. Prove that $V = \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha$. That is, prove that for all x there is an ordinal α so that $x \in V_\alpha$. [Hint: let $P(x)$ be the property "there is an ordinal α so that $x \in V_\alpha$".]

The following theorem schema expresses that it is valid to define something by recursion on a well-founded relation.

Theorem Schema. *Let R be a well-founded relation with domain A. And suppose $P(x,y,z)$ is a function-like property. (That is, for each x,y there is a unique z so that $P(x,y,z)$.) Then there is a function f with domain A so that for each $a \in A$ we have $P(a, f \restriction \{b \in A : b \, R \, a\}, f(a))$.*

Exercise 8 (Reach)*.* Prove this theorem schema, following this outline of steps, which is a generalization of the argument from class that transfinite recursion can be done on well-orders.

(1) Make the following definition of a property $Q(x,y)$:
  - If $x \notin A$, then $Q(x,y)$ if and only if $y = 0$;
  - If $x \in A$, then $Q(x,y)$ if and only if there is a function s with domain $\{b \in A : b \, R \, x\}$ so that (i) for all $b \, R \, x$ we have $P(b, s \restriction \{c \in A : c \, R \, b\}, s(b))$ and (ii) $y = s(x)$.

(2) Show that $Q(x,y)$ is function-like.

(3) Use the Replacement axiom schema on $Q(x,y)$ and A to get a set Y so that for each $a \in A$ there is $y \in Y$ so that $Q(a,y)$.

(4) Define $f \subseteq A \times Y$ as $(a,y) \in f$ if and only if $Q(a,y)$. Conclude that f is the desired function.

One of the most important applications of transfinite recursion on general well-founded relations is the full version of the Mostowski collapse lemma. Say that a binary relation E with domain A is *extensional* if given any $x, y \in A$ we have that $x = y$ if and only if $\{z \in A : z \, E \, x\} = \{z \in A : z \, E \, y\}$. That is, extensional relations are those which satisfy the axiom of extensionality (except formulated with E instead of $\in$).

Theorem (Mostowski)**.** *Let E be a well-founded and extensional binary relation with domain A. Then there is a unique transitive set t and a unique isomorphism $\pi : (A, E) \to (t, \in)$.*

(This is not actually the fullest version of Mostowski's result. There is also a version for properties.)

Exercise 9 (Reach)*.* Prove the Mostowski collapse lemma. [Hint: define, by recursion on E, the function $\pi(x) = \{\pi(y) : y \, E \, x\}$. Show that this π is an isomorphism.]

Exercise 10 (Reach even further)*.* Formulate a theorem schema expressing that transfinite recursion is valid on $\in$. Prove this theorem schema.

It turns out that while transfinite recursion is valid on $\in$, it is not in general valid for all well-founded properties. (Where a property $P(x,y)$ is well-founded if given any nonempty set X there is $m \in X$ so that there is no $x \in X$ with $P(x,m)$.) Proving this, however, requires ideas that go beyond the scope of this class. If you are interested in knowing more, talk to me and I can point you to some references.