MATH454 HOMEWORK 1
DUE THURSDAY, SEPTEMBER 5

The point of this homework assignment is to see how we can construct different number systems starting from just the natural numbers. Along the way we will review some material about equivalence relations. Later in the class we will see how we can construct the natural numbers just using pure sets.

Let me remind you that exercises marked as reach exercises are optional, more difficult problems, or problems that rely on material outside this class and its pre-requisites. You are encouraged to attempt them, but I won't penalize your grade if you don't.

We will do these constructions in stages, building on the previous work. We start by seeing how to represent integers using natural numbers.

Definition. *Let (a, b) and (c, d) be pairs of natural numbers. Say that $(a, b) \sim_{\mathbb{Z}} (c, d)$ if $a+d = b+c$. Given these pairs, define a sum operation as $(a, b) +_{\mathbb{Z}} (c, d) = (a + c, b + d)$ and a multiplication operation as $(a, b) \times_{\mathbb{Z}} (c, d) = (ac + bd, ad + bc)$.*

Recall that an equivalence relation is a relation which is reflexive, symmetric, and transitive. That is, $\sim$ is an equivalence relation if $x \sim x$, $x \sim y$ implies $y \sim x$, and $x \sim y$ and $y \sim z$ implies $x \sim z$.

Exercise 1. Show that $\sim_{\mathbb{Z}}$ is an equivalence relation. Show that $+_{\mathbb{Z}}$ and $\times_{\mathbb{Z}}$ are are well-defined on $\sim_{\mathbb{Z}}$-equivalence classes. That is, show that if $(a, b) \sim_{\mathbb{Z}} (a', b')$ and $(c, d) \sim_{\mathbb{Z}} (c', d')$ then $(a, b) +_{\mathbb{Z}} (c, d) \sim_{\mathbb{Z}} (a', b') +_{\mathbb{Z}} (c', d')$ and $(a, b) \times_{\mathbb{Z}} (c, d) \sim_{\mathbb{Z}} (a', b') \times_{\mathbb{Z}} (c', d')$.

[Hint: Think of (a, b) as representing the integer $a - b$.]

Recall that if $\sim$ is an equivalence relation on a set X then $X/\sim$ is the set of equivalence classes of $\sim$. That is, elements of $X/\sim$ are of the form $[a] = \{x \in X : a \sim x\}$, for $a \in X$. Also recall that if an operation $+$ on X is well-defined on $\sim$-equivalence classes, then we may think of $+$ as an operation on $X/\sim$. Namely, we define $[a] + [b]$ as $[a + b]$, and this is independent of the choice of representative.

Exercise 2. Show that $(\mathbb{N}^2/\sim_{\mathbb{Z}}, +_{\mathbb{Z}}, \times_{\mathbb{Z}})$ is isomorphic to $(\mathbb{Z}, +, \times)$.

[Hint: Show that the map $f(a, b) = a - b$ gives an isomorphism. That is, show that f is onto $\mathbb{Z}$, that $f(a, b) = f(c, d)$ if and only if $(a, b) \sim_{\mathbb{Z}} (c, d)$, that $f((a, b) +_{\mathbb{Z}} (c, d)) = f(a, b) + f(c, d)$, and that $f((a, b) \times_{\mathbb{Z}} (c, d)) = f(a, b) \times f(c, d)$.]

You have just shown that we can represent integers as pairs of natural numbers, defining addition and multiplication for integers in terms of addition and multiplication for natural numbers. Now let's do the same thing, going from integers to rational numbers. I will give the definition here in terms of integers, but you can use the representation we just looked at to rewrite it just in terms of natural numbers.

Definition. *We consider here pairs (p, q) of integers where $q \neq 0$. Given two such pairs (p, q) and (r, s), define the following: Say that $(p, q) \sim_{\mathbb{Q}} (r, s)$ if $ps = qr$. And define operations $+_{\mathbb{Q}}$ and $\times_{\mathbb{Q}}$ as $(p, q) +_{\mathbb{Q}} (r, s) = (ps + rq, qs)$ and $(p, q) \times_{\mathbb{Q}} (r, s) = (pr, qs)$.*

Exercise 3. Show that $\sim_{\mathbb{Q}}$ is an equivalence relation. Also show that $+_{\mathbb{Q}}$ and $\times_{\mathbb{Q}}$ are well-defined on $\sim_{\mathbb{Q}}$-equivalence classes.

[Hint: Think of (p, q) as representing the rational number $\frac{p}{q}$.]

Exercise 4. Show that $(\mathbb{Z}/{\sim_{\mathbb{Q}}}, +_{\mathbb{Q}}, \times_{\mathbb{Q}})$ is isomorphic to $(\mathbb{Q}, +, \times)$.

[Hint: Show that the map $f(p, q) = p/q$ gives an isomorphism.]

Exercise 5. Use the above representations of rational numbers as pairs of integers and integers as pairs of natural numbers to give a representation of rational numbers as quadruples of natural numbers. That is, give definitions for $(a, b, c, d) \sim (a', b', c', d')$, $(a, b, c, d) \oplus (a', b', c', d')$, and $(a, b, c, d) \otimes (a', b', c', d')$ so that $\sim$ is an equivalence relation, $\oplus$ and $\otimes$ are well-defined on $\sim$-equivalence classes, and $(\mathbb{N}^4/{\sim}, \oplus, \otimes)$ is isomorphic to $(\mathbb{Q}, +, \times)$. (I am only asking you to write down the definitions, not prove that they have those properties. But if you like, you can also prove this $\ddot\smile$)

The upshot of all this is that we can represent rational numbers and do arithmetic with them just using natural numbers.

Next let's talk about real numbers. This requires a new idea. We will use one due to Richard Dedekind. (The other well-known construction of the reals from the rationals is due to Cantor.)

Definition. *A* left Dedekind cut *is a nonempty set C of rationals which is closed downward, meaning that if $x \in C$ and $y < x$ then $y \in C$, and so that $C \neq \mathbb{Q}$. Given a real number x, define the* left Dedekind cut of x *to be the set* $\mathrm{LDC}(x) = \{q \in \mathbb{Q} : q \leq x\}$.

Exercise 6. Show that for real numbers x and y that $x = y$ if and only if $\mathrm{LDC}(x) = \mathrm{LDC}(y)$.

The next exercise is a reach exercise because it relies on a fact you learn in Math 331, which is not a formal pre-requisite for this course. If you have taken Math 331, you should do this exercise.

Exercise 7 (Reach). Show that if C is a left Dedekind cut then there is a real number x so that $C = \mathrm{LDC}(x)$.

These exercises shows that we can represent real numbers by certain sets of rational numbers. Since we can represent rational numbers with natural numbers, this means that we can represent real numbers with sets of (tuples of) natural numbers.

Exercise 8 (Reach). Give explicit definitions for real number addition and multiplication in terms of left Dedekind cuts. That is, give definitions for operations $\oplus$ and $\otimes$ on left Dedekind cuts so that if $C = \mathrm{LDC}(x)$ and $D = \mathrm{LDC}(y)$ then $\mathrm{LDC}(x + y) = C \oplus D$ and $\mathrm{LDC}(xy) = C \otimes D$. As a further reach, prove that your definitions satisfy these properties.

To close off this homework set, let's address a small issue from the earlier representations. We represented integers as pairs of natural numbers. But we might what to know whether we can represent integers as individual natural numbers. It turns out that the answer is yes, because we can represent pairs of natural numbers as individual natural numbers.

Definition. *The* Cantor pairing function *is the function* $p : \mathbb{N}^2 \to \mathbb{N}$ *defined as*

$$p(a, b) = \frac{(a + b)(a + b + 1)}{2} + b.$$

Exercise 9 (Reach). Show that the Cantor pairing function is a bijection from $\mathbb{N}^2$ to $\mathbb{N}$.

[Hint: Show that given any natural number c there are unique a and b so that $c = p(a, b)$. One way to do this is to pretend you already know that $c = p(a, b)$ and work backward to figure out what a and b must be.]