Uniquely Completable Partial Latin Squares
or
Can we Play Infinite Sudoku?

Matt Ollis

Emerson College, Boston, USA

April 25, 2024

Includes joint work with Aurora Callahan, Emma Hassan, Kaethe Minden and Yolanda Zhu

Sudoku puzzle

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

Source: `https://en.wikipedia.org/wiki/Sudoku`

Sudoku solution

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

Source: `https://en.wikipedia.org/wiki/Sudoku`

Latin squares

Definition

A latin square *of order n is an $n \times n$ array with symbols taken from a set S of size n with each symbol appearing exactly once in each row and exactly once in each column.*

Latin squares

Definition

A latin square *of order n is an $n \times n$ array with symbols taken from a set S of size n with each symbol appearing exactly once in each row and exactly once in each column.*

A latin square of order 6:

0	1	2	3	4	5
1	4	5	2	3	0
2	0	3	5	1	4
3	5	4	1	0	2
4	2	1	0	5	3
5	3	0	4	2	1

Practical applications

Source: “Trap Cropping Harlequin Bug: Distance of Separation Influences Female Movement and Oviposition", Bier *et al.* in the *Journal of Economic Entomology* (2021).

How many latin squares are there?

- $n = 2$: 1
- $n = 3$: 1
- $n = 4$: 4

How many latin squares are there?

- $n = 2$: 1
- $n = 3$: 1
- $n = 4$: 4
- $n = 5$: 56

How many latin squares are there?

- $n = 2$: 1
- $n = 3$: 1
- $n = 4$: 4
- $n = 5$: 56
- $n = 6$: 9,408

How many latin squares are there?

- $n = 2$: 1
- $n = 3$: 1
- $n = 4$: 4
- $n = 5$: 56
- $n = 6$: 9,408
- $n = 7$: 16,942,080
- $n = 8$: 535,281,401,856
- $n = 9$: 377,597,570,964,258,816
- $n = 10$: 7,580,721,483,160,132,811,489,280
- $n = 11$: 5,363,937,773,277,371,298,119,673,540,771,840

How many latin squares are there?

- $n = 2$: 1
- $n = 3$: 1
- $n = 4$: 4
- $n = 5$: 56
- $n = 6$: 9,408
- $n = 7$: 16,942,080
- $n = 8$: 535,281,401,856
- $n = 9$: 377,597,570,964,258,816
- $n = 10$: 7,580,721,483,160,132,811,489,280
- $n = 11$: 5,363,937,773,277,371,298,119,673,540,771,840

The number of latin squares of order n is at least $\frac{(n!)^{2n}}{n^{n^2}}$.

How many latin squares? (Prime factors version)

- $n = 2$: 1
- $n = 3$: 1
- $n = 4$: 2^2
- $n = 5$: $2^3 \cdot 7$
- $n = 6$: $2^6 \cdot 3 \cdot 7^2$
- $n = 7$: $2^{10} \cdot 3 \cdot 5 \cdot 1103$
- $n = 8$: $2^{17} \cdot 3 \cdot 1361291$
- $n = 9$: $2^{21} \cdot 3^2 \cdot 5231 \cdot 3824477$
- $n = 10$: $2^{28} \cdot 3^2 \cdot 5 \cdot 31 \cdot 37 \cdot 547135293937$
- $n = 11$: $2^{35} \cdot 3^4 \cdot 5 \cdot 2801 \cdot 2206499 \cdot 62368028479$

Partial latin squares

Definition

A partial latin square of order n is an $n \times n$ array with symbols taken from a set S of size n with each symbol appearing at most once in each row and at most once in each column.

Partial latin squares

Definition

A partial latin square *of order n is an $n \times n$ array with symbols taken from a set S of size n with each symbol appearing* at most *once in each row and* at most *once in each column.*

A partial latin square of order 6:

$$\begin{array}{cccccc}
0 & 1 & \cdot & 3 & \cdot & \cdot \\
\cdot & \cdot & 5 & \cdot & 3 & 0 \\
\cdot & 0 & 3 & \cdot & \cdot & \cdot \\
3 & \cdot & \cdot & \cdot & 0 & 2 \\
\cdot & 2 & \cdot & \cdot & \cdot & 3 \\
\cdot & 3 & \cdot & \cdot & 2 & 1
\end{array}$$

Completability

Definition

If it is possible to fill in the empty cells in a partial latin square P to obtain a latin square L, then P is completable to L.

Completability

Definition

If it is possible to fill in the empty cells in a partial latin square P to obtain a latin square L, then P is completable *to L.*

The partial latin square of order 6 from the previous slide is completable to the latin square example we saw earlier.

Completability

Definition

If it is possible to fill in the empty cells in a partial latin square P to obtain a latin square L, then P is completable *to L.*

The partial latin square of order 6 from the previous slide is completable to the latin square example we saw earlier.

Not all partial latin squares are completable:

$$\begin{matrix} 0 & 1 & 2 & 3 & 4 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 5 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$$

Unique completability

Definition

If a partial latin square is completable to exactly one latin square, then it is uniquely completable.

Unique completability

Definition

If a partial latin square is completable to exactly one latin square, then it is uniquely completable.

Not all completable latin squares are uniquely completable:

0	1	2	3	4	5
1	4	5	2	3	0
2	0	3	5	1	4
3	5	4	1	0	2
4	2	1	0	5	3
5	3	0	4	2	1

0	1	2	3	4	5
1	4	5	2	3	0
2	0	3	5	1	4
3	5	1	4	0	2
4	2	0	1	5	3
5	3	4	0	2	1

Critical sets

Definition

Let P be a uniquely completable partial latin square. Then P is critical *if removing any single entry means that it is no longer uniquely completable.*

Critical sets

Definition

Let P be a uniquely completable partial latin square. Then P is critical if removing any single entry means that it is no longer uniquely completable.

A critical set of order 6:

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdot \\ 1 & 2 & 3 & 4 & \cdot & \cdot \\ 2 & 3 & 4 & \cdot & \cdot & \cdot \\ 3 & 4 & \cdot & \cdot & \cdot & \cdot \\ 4 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

Critical sets

Definition

Let P be a uniquely completable partial latin square. Then P is critical if removing any single entry means that it is no longer uniquely completable.

A critical set of order 6:

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdot \\ 1 & 2 & 3 & 4 & \cdot & \cdot \\ 2 & 3 & 4 & \cdot & \cdot & \cdot \\ 3 & 4 & \cdot & \cdot & \cdot & \cdot \\ 4 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 5 & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

Critical sets

Definition

Let P be a uniquely completable partial latin square. Then P is critical if removing any single entry means that it is no longer uniquely completable.

A critical set of order 6:

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & \cdot \\ 2 & 3 & 4 & 5 & \cdot & \cdot \\ 3 & 4 & 5 & \cdot & \cdot & \cdot \\ 4 & 5 & \cdot & \cdot & \cdot & \cdot \\ 5 & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

Critical sets

Definition

Let P be a uniquely completable partial latin square. Then P is critical if removing any single entry means that it is no longer uniquely completable.

A critical set of order 6:

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 0 \\ 2 & 3 & 4 & 5 & 0 & \cdot \\ 3 & 4 & 5 & 0 & \cdot & \cdot \\ 4 & 5 & 0 & \cdot & \cdot & \cdot \\ 5 & 0 & \cdot & \cdot & \cdot & \cdot \end{array}$$

Critical sets

Definition

Let P be a uniquely completable partial latin square. Then P is critical if removing any single entry means that it is no longer uniquely completable.

A critical set of order 6:

0	1	2	3	4	5
1	2	3	4	5	0
2	3	4	5	0	1
3	4	5	0	1	2
4	5	0	1	2	3
5	0	1	2	3	4

Critical sets

Definition

Let P be a uniquely completable partial latin square. Then P is critical if removing any single entry means that it is no longer uniquely completable.

A critical set of order 6:

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdot \\ 1 & 2 & \cdot & 4 & \cdot & \cdot \\ 2 & 3 & 4 & \cdot & \cdot & \cdot \\ 3 & 4 & \cdot & \cdot & \cdot & \cdot \\ 4 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

Critical sets

Definition

Let P be a uniquely completable partial latin square. Then P is critical if removing any single entry means that it is no longer uniquely completable.

A critical set of order 6:

0	1	2	3	4	5
1	2	3	4	5	0
2	3	4	5	0	1
3	4	5	0	1	2
4	5	0	1	2	3
5	0	1	2	3	4

Critical sets

Definition

Let P be a uniquely completable partial latin square. Then P is critical if removing any single entry means that it is no longer uniquely completable.

A critical set of order 6:

0	1	2	3	4	5
1	2	0	4	5	3
2	3	4	5	0	1
3	4	5	0	1	2
4	5	3	1	2	0
5	0	1	2	3	4

Critical sets

Definition

Let P be a uniquely completable partial latin square. Then P is critical *if removing any single entry means that it is no longer uniquely completable.*

A critical set of order 6:

$$\begin{array}{cccccc} 0 & \cdot & 2 & 3 & 4 & \cdot \\ 1 & 2 & 3 & 4 & \cdot & \cdot \\ 2 & 3 & 4 & \cdot & \cdot & \cdot \\ 3 & 4 & \cdot & \cdot & \cdot & \cdot \\ 4 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

Critical sets

Definition

Let P be a uniquely completable partial latin square. Then P is critical if removing any single entry means that it is no longer uniquely completable.

A critical set of order 6:

0	1	2	3	4	5
1	2	3	4	5	0
2	3	4	5	0	1
3	4	5	0	1	2
4	5	0	1	2	3
5	0	1	2	3	4

Critical sets

Definition

Let P be a uniquely completable partial latin square. Then P is critical if removing any single entry means that it is no longer uniquely completable.

A critical set of order 6:

0	5	2	3	4	1
1	2	3	4	5	0
2	3	4	1	0	5
3	4	5	0	1	2
4	1	0	5	2	3
5	0	1	2	3	4

Trades

Formalising the method from the previous slides:

Definition

Let L and L' be distinct latin squares and let $T \subseteq L$ and $T' \subseteq L'$ with $T \cap T' = \emptyset$. If $L \setminus T = L' \setminus T'$ then T is a trade.

Trades

Formalising the method from the previous slides:

Definition

Let L and L' be distinct latin squares and let $T \subseteq L$ and $T' \subseteq L'$ with $T \cap T' = \emptyset$. If $L \setminus T = L' \setminus T'$ then T is a trade.

Lemma

Let P be a partial latin square contained in L and let T be a trade. If $T \cap P = \emptyset$ then P is not uniquely completable.

Trades

Formalising the method from the previous slides:

Definition

Let L and L' be distinct latin squares and let $T \subseteq L$ and $T' \subseteq L'$ with $T \cap T' = \emptyset$. If $L \setminus T = L' \setminus T'$ then T is a trade*.*

Lemma

Let P be a partial latin square contained in L and let T be a trade. If $T \cap P = \emptyset$ then P is not uniquely completable.

Lemma

Let P be uniquely completeable to L. If for each $e \in P$ there is a trade T with $T \cap P = \{e\}$ then P is critical.

Density

Definition

Let P be a uniquely completable partial latin square of order n with t entries. The density *of P is $\rho = t/n^2$.*

Density

Definition

Let P be a uniquely completable partial latin square of order n with t entries. The density *of P is $\rho = t/n^2$.*

Critical sets of order 6 with densities $1/4$ and $5/12$:

0	1	2	·	·	·
1	2	·	·	·	·
2	·	·	·	·	·
·	·	·	·	·	·
·	·	·	·	·	3
·	·	·	·	3	4

0	1	2	3	4	·
1	2	3	4	·	·
2	3	4	·	·	·
3	4	·	·	·	·
4	·	·	·	·	·
·	·	·	·	·	·

Density

Definition

Let P be a uniquely completable partial latin square of order n with t entries. The density *of P is $\rho = t/n^2$.*

Critical sets of order 6 with densities $1/4$ and $5/12$:

0	1	2	·	·	·
1	2	·	·	·	·
2	·	·	·	·	·
·	·	·	·	·	·
·	·	·	·	·	3
·	·	·	·	3	4

0	1	2	3	4	·
1	2	3	4	·	·
2	3	4	·	·	·
3	4	·	·	·	·
4	·	·	·	·	·
·	·	·	·	·	·

These partial latin squares both complete to the *back-circulant* latin square of order 6. In general denote the back-circulant square of order n, which is also the addition table for the integers modulo n, by L_n.

More density

Theorem

L_n has critical sets of densities $(n-1)/2n$ and approximately $1/4$.

More density

Theorem

L_n has critical sets of densities $(n-1)/2n$ and approximately $1/4$.

Conjecture (Mahmoodian, 1995; Bate and van Rees, 1999)

The minimum possible density of a critical set is approximately $1/4$.
The maximum possible density of a critical set of L_n is $(n-1)/2n$.

More density

Theorem

L_n has critical sets of densities $(n-1)/2n$ and approximately $1/4$.

Conjecture (Mahmoodian, 1995; Bate and van Rees, 1999)

The minimum possible density of a critical set is approximately $1/4$.
The maximum possible density of a critical set of L_n is $(n-1)/2n$.

Theorem (Hatami and Qian, 2018)

For sufficiently large n, every critical set has density at least $1/10000$.

Partitions into critical sets

A partition of a latin square of order 6 into three disjoint critical sets:

0	1	2	3	4	5
1	0	3	2	5	4
2	3	4	5	0	1
3	4	5	1	2	0
4	5	1	0	3	4
5	2	0	4	1	3

Partitions into critical sets

A partition of a latin square of order 6 into three disjoint critical sets:

0	1	2	3	4	5
1	0	3	2	5	4
2	3	4	5	0	1
3	4	5	1	2	0
4	5	1	0	3	4
5	2	0	4	1	3

Theorem (Adams, Bean and Khodkar 2001)

A partition of a latin square of order n into k parts exists when:

- $k = 4$, *for all* n. *Densities are all approximately* $1/4$.
- $k = 3$, *for* $n = 4, 5, 6$. *Densities between* $5/18$ *and* $7/18$.
- $k = 2$, *for* $n = 8$. *Densities are* $1/2$.

Back to sudoku

Theorem (McGuire, Tufemgann and Civario, 2014)

The smallest critical set for a 9×9 *sudoku has 17 filled cells.*

			8		1			
							4	3
5								
				7		8		
						1		
	2			3				
6							7	5
		3	4					
			2			6		

Paths to fame and fortune (Part 1)

- Why is the number of latin squares divisible by a high power of 2? Good first step: Is it always divisible by 16 for $n \geq 6$?

Paths to fame and fortune (Part 1)

- Why is the number of latin squares divisible by a high power of 2? Good first step: Is it always divisible by 16 for $n \geq 6$?
- Close the gap between $1/10000$ and $1/4$ for minimum possible density of a critical square. A graph theoretic conjecture would, if true, increase $1/10000$ to $1/128$.

Paths to fame and fortune (Part 1)

- Why is the number of latin squares divisible by a high power of 2? Good first step: Is it always divisible by 16 for $n \geq 6$?
- Close the gap between $1/10000$ and $1/4$ for minimum possible density of a critical square. A graph theoretic conjecture would, if true, increase $1/10000$ to $1/128$.
- Find more partitions of latin squares into critical sets. Possibly good first step: Partition the Cayley table of $\mathbb{Z}_2^4$ into two critical sets of density $1/2$.

Paths to fame and fortune (Part 1)

- Why is the number of latin squares divisible by a high power of 2? Good first step: Is it always divisible by 16 for $n \geq 6$?
- Close the gap between $1/10000$ and $1/4$ for minimum possible density of a critical square. A graph theoretic conjecture would, if true, increase $1/10000$ to $1/128$.
- Find more partitions of latin squares into critical sets. Possibly good first step: Partition the Cayley table of $\mathbb{Z}_2^4$ into two critical sets of density $1/2$.
- Come up with a good reason why a 16-clue sudoku of order 9 cannot exist. Use this to find results for sudoku of order m^2.

Paths to fame and fortune (Part 1)

- Why is the number of latin squares divisible by a high power of 2? Good first step: Is it always divisible by 16 for $n \geq 6$?
- Close the gap between $1/10000$ and $1/4$ for minimum possible density of a critical square. A graph theoretic conjecture would, if true, increase $1/10000$ to $1/128$.
- Find more partitions of latin squares into critical sets. Possibly good first step: Partition the Cayley table of $\mathbb{Z}_2^4$ into two critical sets of density $1/2$.
- Come up with a good reason why a 16-clue sudoku of order 9 cannot exist. Use this to find results for sudoku of order m^2.
- Add other constraints. What if we insist the square is the Cayley table of a group (but not which group)? What if the entries must satisfy some neighbour-pair properties?

Paths to fame and fortune (Part 1)

- Why is the number of latin squares divisible by a high power of 2? Good first step: Is it always divisible by 16 for $n \geq 6$?
- Close the gap between 1/10000 and 1/4 for minimum possible density of a critical square. A graph theoretic conjecture would, if true, increase 1/10000 to 1/128.
- Find more partitions of latin squares into critical sets. Possibly good first step: Partition the Cayley table of $\mathbb{Z}_2^4$ into two critical sets of density 1/2.
- Come up with a good reason why a 16-clue sudoku of order 9 cannot exist. Use this to find results for sudoku of order m^2.
- Add other constraints. What if we insist the square is the Cayley table of a group (but not which group)? What if the entries must satisfy some neighbour-pair properties?

Disclaimer: Like the latin squares, both the fame and the fortune available are *very* finite.

Infinite latin squares

Definition

An infinite latin square *on* $\mathbb{Z}$ *is an assignment of integers to all of the points of the lattice* $\mathbb{Z}^2$ *such that every integer appears exactly once in each row and exactly once in each column.*

The definitions of *partial* latin squares, *unique completability* and *critical set* apply exactly as in the finite case.

Infinite latin squares

Definition

An infinite latin square on $\mathbb{Z}$ is an assignment of integers to all of the points of the lattice $\mathbb{Z}^2$ such that every integer appears exactly once in each row and exactly once in each column.

The definitions of *partial* latin squares, *unique completability* and *critical set* apply exactly as in the finite case. *Density* needs a new definition but it works as you would probably expect (except when it doesn't work).

Infinite latin squares

Definition

An infinite latin square on $\mathbb{Z}$ is an assignment of integers to all of the points of the lattice $\mathbb{Z}^2$ such that every integer appears exactly once in each row and exactly once in each column.

The definitions of *partial* latin squares, *unique completability* and *critical set* apply exactly as in the finite case. *Density* needs a new definition but it works as you would probably expect (except when it doesn't work).

We focus on a particular square, the *integer addition square*:

$$L_{\mathbb{Z}} = \{(x, y, x + y) : x, y \in \mathbb{Z}\}$$

That is, the point at "cell" at position (x, y) has entry $x + y$.

A critical set in $L_{\mathbb{Z}}$

$$
\begin{array}{ccccccccccc}
 & & & & & \vdots & & & & & \iddots \\
 & \cdot & \cdot & \cdot & \cdot & 4 & 5 & 6 & 7 & 8 & \\
 & \cdot & \cdot & \cdot & \cdot & 3 & 4 & 5 & 6 & 7 & \\
 & \cdot & \cdot & \cdot & \cdot & 2 & 3 & 4 & 5 & 6 & \\
 & \cdot & \cdot & \cdot & \cdot & 1 & 2 & 3 & 4 & 5 & \cdots \\
\cdots & -4 & -3 & -2 & -1 & \odot & \cdot & \cdot & \cdot & \cdot & \\
 & -5 & -4 & -3 & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -6 & -5 & -4 & -3 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -7 & -6 & -5 & -4 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -8 & -7 & -6 & -5 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\iddots & & & & \vdots & & & & & &
\end{array}
$$

A critical set in $L_{\mathbb{Z}}$

$$
\begin{array}{ccccccccccc}
 & & & & & \vdots & & & & & \iddots \\
 & \cdot & \cdot & \cdot & \cdot & 4 & 5 & 6 & 7 & 8 & \\
 & \cdot & \cdot & \cdot & \cdot & 3 & 4 & 5 & 6 & 7 & \\
 & \cdot & \cdot & \cdot & \cdot & 2 & 3 & 4 & 5 & 6 & \\
 & \cdot & \cdot & \cdot & \cdot & 1 & 2 & 3 & 4 & 5 & \cdots \\
\cdots & -4 & -3 & -2 & -1 & \color{red}{0} & \cdot & \cdot & \cdot & \cdot & \\
 & -5 & -4 & -3 & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -6 & -5 & -4 & -3 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -7 & -6 & -5 & -4 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -8 & -7 & -6 & -5 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\iddots & & & & \vdots & & & & & &
\end{array}
$$

A critical set in $L_{\mathbb{Z}}$

$$
\begin{array}{ccccccccccc}
 & & & & & \vdots & & & & & \iddots \\
 & \cdot & \cdot & \cdot & \cdot & 4 & 5 & 6 & 7 & 8 & \\
 & \cdot & \cdot & \cdot & \cdot & 3 & 4 & 5 & 6 & 7 & \\
 & \cdot & \cdot & \cdot & \cdot & 2 & 3 & 4 & 5 & 6 & \\
 & \cdot & \cdot & \cdot & \cdot & 1 & 2 & 3 & 4 & 5 & \cdots \\
\cdots & -4 & -3 & -2 & -1 & \color{red}{0} & \color{red}{1} & \cdot & \cdot & \cdot & \\
 & -5 & -4 & -3 & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -6 & -5 & -4 & -3 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -7 & -6 & -5 & -4 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -8 & -7 & -6 & -5 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\iddots & & & & \vdots & & & & & &
\end{array}
$$

A critical set in $L_{\mathbb{Z}}$

$$
\begin{array}{ccccccccccc}
 & & & & & \vdots & & & & & \iddots \\
 & \cdot & \cdot & \cdot & \cdot & 4 & 5 & 6 & 7 & 8 & \\
 & \cdot & \cdot & \cdot & \cdot & 3 & 4 & 5 & 6 & 7 & \\
 & \cdot & \cdot & \cdot & \cdot & 2 & 3 & 4 & 5 & 6 & \\
 & \cdot & \cdot & \cdot & \cdot & 1 & 2 & 3 & 4 & 5 & \cdots \\
\cdots & -4 & -3 & -2 & -1 & \color{red}{0} & \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{\cdots} \\
 & -5 & -4 & -3 & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -6 & -5 & -4 & -3 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -7 & -6 & -5 & -4 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -8 & -7 & -6 & -5 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\iddots & & & & \vdots & & & & & &
\end{array}
$$

A critical set in $L_{\mathbb{Z}}$

$$
\begin{array}{ccccccccccc}
 & & & & & \vdots & & & & & \iddots \\
 & \cdot & \cdot & \cdot & \cdot & 4 & 5 & 6 & 7 & 8 & \\
 & \cdot & \cdot & \cdot & \cdot & 3 & 4 & 5 & 6 & 7 & \\
 & \cdot & \cdot & \cdot & \cdot & 2 & 3 & 4 & 5 & 6 & \\
 & \cdot & \cdot & \cdot & \cdot & 1 & 2 & 3 & 4 & 5 & \cdots \\
\cdots & -4 & -3 & -2 & -1 & \color{red}{0} & \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{\cdots} \\
 & -5 & -4 & -3 & -2 & \color{red}{-1} & \color{red}{0} & \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{\cdots} \\
 & -6 & -5 & -4 & -3 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -7 & -6 & -5 & -4 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -8 & -7 & -6 & -5 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\iddots & & & & \vdots & & & & & &
\end{array}
$$

A critical set in $L_{\mathbb{Z}}$

$$
\begin{array}{ccccccccccc}
 & & & & & \vdots & & & & & \iddots \\
 & \cdot & \cdot & \cdot & \cdot & 4 & 5 & 6 & 7 & 8 & \\
 & \cdot & \cdot & \cdot & \cdot & 3 & 4 & 5 & 6 & 7 & \\
 & \cdot & \cdot & \cdot & \cdot & 2 & 3 & 4 & 5 & 6 & \\
 & \cdot & \cdot & \cdot & \cdot & 1 & 2 & 3 & 4 & 5 & \cdots \\
\cdots & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & \cdots \\
 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \cdots \\
 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & \\
 & -7 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & \\
 & -8 & -7 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & \\
\iddots & & & & \vdots & & & \vdots & & &
\end{array}
$$

A critical set in $L_{\mathbb{Z}}$

$$
\begin{array}{ccccccccccc}
 & & & \vdots & & \vdots & & & & & \iddots \\
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\
 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\
 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \\
\cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & \cdots \\
\cdots & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & \cdots \\
 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \cdots \\
 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & \\
 & -7 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & \\
 & -8 & -7 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & \\
\iddots & & & & \vdots & & & \vdots & & &
\end{array}
$$

A critical set in $L_{\mathbb{Z}}$

$$
\begin{array}{ccccccccccc}
 & & & & & \vdots & & & & & \iddots \\
 & \cdot & \cdot & \cdot & \cdot & 4 & 5 & 6 & 7 & 8 & \\
 & \cdot & \cdot & \cdot & \cdot & 3 & 4 & 5 & 6 & 7 & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & 3 & 4 & 5 & 6 & \\
 & \cdot & \cdot & \cdot & \cdot & 1 & 2 & 3 & 4 & 5 & \cdots \\
\cdots & -4 & -3 & -2 & -1 & \odot & \cdot & \cdot & \cdot & \cdot & \\
 & -5 & -4 & -3 & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -6 & -5 & -4 & -3 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -7 & -6 & -5 & -4 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -8 & -7 & -6 & -5 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\iddots & & & & \vdots & & & & & &
\end{array}
$$

A critical set in $L_{\mathbb{Z}}$

$$
\begin{array}{ccccccccccc}
 & & & \vdots & & \vdots & & & & & \iddots \\
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\
 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\
 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \\
\cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & \cdots \\
\cdots & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & \cdots \\
 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \cdots \\
 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & \\
 & -7 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & \\
 & -8 & -7 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & \\
\iddots & & & & \vdots & & & \vdots & & &
\end{array}
$$

A critical set in $L_{\mathbb{Z}}$

$$
\begin{array}{ccccccccccc}
 & & & \vdots & & \vdots & & & & & \iddots \\
 & 2 & 1 & 0 & 3 & 4 & 5 & 6 & 7 & 8 & \\
 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\
 & -2 & -1 & 2 & 1 & 0 & 3 & 4 & 5 & 6 & \\
\cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & \cdots \\
\cdots & -4 & -3 & -2 & -1 & 2 & 1 & 0 & 3 & 4 & \cdots \\
 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \cdots \\
 & -6 & -5 & -4 & -3 & -2 & -1 & 2 & 1 & 0 & \\
 & -7 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & \\
 & -8 & -7 & -6 & -5 & -4 & -3 & -2 & -1 & 2 & \\
\iddots & & & & \vdots & & & \vdots & & &
\end{array}
$$

Another critical set in $L_{\mathbb{Z}}$

$$
\begin{array}{ccccccccccccc}
\ddots & & & & & & & & & & & & \\
 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -2 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -3 & -2 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -4 & -3 & -2 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdots & -5 & -4 & -3 & -2 & -1 & \odot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & 2 & 3 & 4 & \cdots \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & 2 & 3 & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & 2 & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \\
 & & & & & & & & & & & & \ddots
\end{array}
$$

Another critical set in $L_{\mathbb{Z}}$

$$
\begin{array}{ccccccccccccc}
\ddots & & & & & & & & & & & & \\
 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -2 & -1 & \cdot & \cdot & \cdot & 3 & \cdot & 5 & \cdot & \cdot & \cdot & \\
 & -3 & -2 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -4 & -3 & -2 & -1 & \cdot & 1 & \cdot & 3 & \cdot & \cdot & \cdot & \\
\cdots & -5 & -4 & -3 & -2 & -1 & \odot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 0 & 1 & 2 & 3 & 4 & \cdots \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & 2 & 3 & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & -3 & \cdot & -1 & 0 & 1 & 2 & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & -5 & \cdot & -3 & \cdot & \cdot & 0 & \\
 & & & & & & & & & & & & \ddots
\end{array}
$$

Another critical set in $L_{\mathbb{Z}}$

$$
\begin{array}{ccccccccccccc}
\ddots & & & & & & & & & & & & \\
 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -2 & -1 & \cdot & \cdot & \cdot & \color{blue}{5} & \cdot & \color{blue}{3} & \cdot & \cdot & \cdot & \\
 & -3 & -2 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & -4 & -3 & -2 & -1 & \cdot & \color{blue}{3} & \cdot & \color{blue}{1} & \cdot & \cdot & \cdot & \\
\cdots & -5 & -4 & -3 & -2 & -1 & \odot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \color{blue}{1} & 0 & \color{blue}{-1} & 2 & 3 & 4 & \cdots \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & 2 & 3 & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \color{blue}{-1} & \cdot & \color{blue}{-3} & 0 & 1 & 2 & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \color{blue}{-3} & \cdot & \color{blue}{-5} & \cdot & \cdot & 0 & \\
 & & & & & & & & & & & & \ddots
\end{array}
$$

Something weird

Theorem

There is a uniquely completable partial square in $L_{\mathbb{Z}}$ that contains no critical set.

Something weird

Theorem

There is a uniquely completable partial square in $L_{\mathbb{Z}}$ that contains no critical set.

$$
\begin{array}{ccccccccccccc}
 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & 5 & \cdot & \cdot & \cdot & \cdot & \\
 & -2 & -1 & \cdot & \cdot & \cdot & \cdot & 4 & \cdot & \cdot & \cdot & \cdot & \\
 & -3 & -2 & -1 & \cdot & \cdot & \cdot & 3 & \cdot & \cdot & \cdot & \cdot & \\
 & -4 & -3 & -2 & -1 & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \\
\cdots & -5 & -4 & -3 & -2 & -1 & \odot & 1 & \cdot & \cdot & \cdot & \cdot & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 2 & 3 & 4 & \cdots \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & 2 & 3 & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & 2 & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \\
 & & & & & & & & & & & & \ddots
\end{array}
$$

Something familiar

Theorem

$L_{\mathbb{Z}}$ can be partitioned into three critical sets, one with density $1/2$ and two with density $1/4$.

Something familiar

Theorem

$L_{\mathbb{Z}}$ can be partitioned into three critical sets, one with density $1/2$ and two with density $1/4$.

			⋮		⋮					⋰
	0	1	2	3	4	5	6	7	8	
	−1	0	1	2	3	4	5	6	7	
	−2	−1	0	1	2	3	4	5	6	
⋯	−3	−2	−1	0	1	2	3	4	5	⋯
⋯	−4	−3	−2	−1	0	1	2	3	4	⋯
	−5	−4	−3	−2	−1	0	1	2	3	⋯
	−6	−5	−4	−3	−2	−1	0	1	2	
	−7	−6	−5	−4	−3	−2	−1	0	1	
⋰				⋮			⋮			

Something new

Theorem

$L_{\mathbb{Z}}$ has infinitely many critical sets with density greater than $1/2$. The largest density critical set we have constructed has density $95/176$.

Something new

Theorem

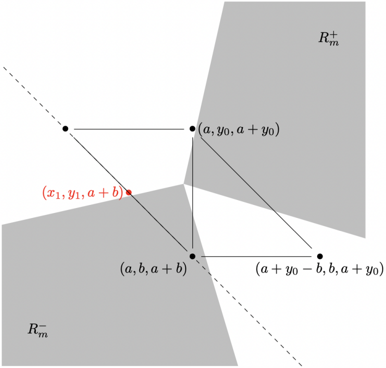
$L_{\mathbb{Z}}$ has infinitely many critical sets with density greater than $1/2$. The largest density critical set we have constructed has density $95/176$.

	·	·	·	·	·	·	·	·	11	12	13	14	15	
	·	·	·	·	·	·	·	·	10	11	12	13	14	
	·	·	·	·	·	·	·	·	9	10	11	12	13	
	·	·	·	·	·	·	·	·	8	9	10	11	12	
	·	·	·	·	·	·	·	6	7	8	9	10	11	
	·	·	·	·	·	·	·	5	6	7	8	9	10	
	·	·	·	·	·	·	·	4	5	6	7	8	9	
	·	·	·	·	·	·	·	3	4	5	6	7	8	
	·	·	·	·	·	·	·	2	3	4	5	6	7	
$\cdots$	·	·	·	·	·	·	$\odot$	1	2	3	4	5	6	$\cdots$
	·	−6	−5	−4	−3	−2	−1	·	·	·	3	4	5	
	−8	−7	−6	−5	−4	−3	−2	·	·	·	·	·	·	
	−9	−8	−7	−6	−5	−4	−3	·	·	·	·	·	·	
	−10	−9	−8	−7	−6	−5	−4	−3	·	·	·	·	·	
	−11	−10	−9	−8	−7	−6	−5	−4	·	·	·	·	·	
	−12	−11	−10	−9	−8	−7	−6	−5	·	·	·	·	·	
	−13	−12	−11	−10	−9	−8	−7	−6	−5	·	·	·	·	
	−14	−13	−12	−11	−10	−9	−8	−7	−6	·	·	·	·	
	−15	−14	−13	−12	−11	−10	−9	−8	−7	·	·	·	·	
	−16	−15	−14	−13	−12	−11	−10	−9	−8	−7	·	·	·	
	−17	−16	−15	−14	−13	−12	−11	−10	−9	−8	·	·	·	
	−18	−17	−16	−15	−14	−13	−12	−11	−10	−9	·	·	·	
	−19	−18	−17	−16	−15	−14	−13	−12	−11	−10	−9	·	·	
$\iddots$							$\vdots$							$\ddots$

Something new

Theorem

$L_{\mathbb{Z}}$ has infinitely many critical sets with density greater than $1/2$. The largest density critical set we have constructed has density $95/176$.



Something new (ctd.)

Let. $M = m^2 - m - 1$. The critical squares are given by:

$$R_m^+ = \left\{(x, y, x+y) : x > 0 \text{ and } -\tfrac{1}{m}x < y \leq Mx\right\} \subseteq L_{\mathbb{Z}},$$

$$R_m^- = \left\{(x, y, x+y) : y < 0 \text{ and } My \leq x < -\tfrac{1}{m}y\right\} \subseteq L_{\mathbb{Z}}.$$

Set $R_m = R_m^+ \cup R_m^-$.

Something new (ctd.)

Let. $M = m^2 - m - 1$. The critical squares are given by:

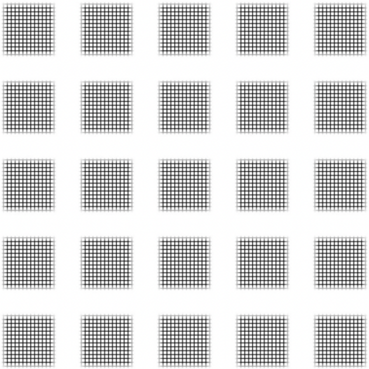
$$R_m^+ = \left\{(x, y, x+y) : x > 0 \text{ and } -\tfrac{1}{m}x < y \leq Mx\right\} \subseteq L_{\mathbb{Z}},$$

$$R_m^- = \left\{(x, y, x+y) : y < 0 \text{ and } My \leq x < -\tfrac{1}{m}y\right\} \subseteq L_{\mathbb{Z}}.$$

Set $R_m = R_m^+ \cup R_m^-$.

The density of R_m is $\frac{2m^3-m^2-4m-1}{4m^3-4m^2-4m}$.

Infinite Sudoku



Infinite Sudoku
The $\mathbb{Z}$-Sudoku board is a
$\mathbb{Z} \times \mathbb{Z}$ array of $\mathbb{Z} \times \mathbb{Z}$ sub-boards

From: Infinite Sudoku and the Sudoku Game (J. D. Hamkins),

`https://jdh.hamkins.org/infinite-sudoku-and-the-sudoku-game/`

Paths to fame and fortune (Part 2)

- Find more critical sets in $L_{\mathbb{Z}}$, especially with densities other than $1/4$, $3/8$, $1/2$ and the values we already have up to $95/176$. In particular, what is the smallest density for a critical set of $L_{\mathbb{Z}}$.

Paths to fame and fortune (Part 2)

- Find more critical sets in $L_{\mathbb{Z}}$, especially with densities other than $1/4$, $3/8$, $1/2$ and the values we already have up to $95/176$. In particular, what is the smallest density for a critical set of $L_{\mathbb{Z}}$.
- Same question, but for other infinite latin squares.

Paths to fame and fortune (Part 2)

- Find more critical sets in $L_{\mathbb{Z}}$, especially with densities other than $1/4$, $3/8$, $1/2$ and the values we already have up to $95/176$. In particular, what is the smallest density for a critical set of $L_{\mathbb{Z}}$.
- Same question, but for other infinite latin squares.
- What happens if we replace $\mathbb{Z}$ with $\mathbb{R}$? Or some other cardinality set?

Paths to fame and fortune (Part 2)

- Find more critical sets in $L_{\mathbb{Z}}$, especially with densities other than $1/4$, $3/8$, $1/2$ and the values we already have up to $95/176$. In particular, what is the smallest density for a critical set of $L_{\mathbb{Z}}$.
- Same question, but for other infinite latin squares.
- What happens if we replace $\mathbb{Z}$ with $\mathbb{R}$? Or some other cardinality set?
- Is there a critical set with undefined density?

Paths to fame and fortune (Part 2)

- Find more critical sets in $L_{\mathbb{Z}}$, especially with densities other than $1/4$, $3/8$, $1/2$ and the values we already have up to $95/176$. In particular, what is the smallest density for a critical set of $L_{\mathbb{Z}}$.
- Same question, but for other infinite latin squares.
- What happens if we replace $\mathbb{Z}$ with $\mathbb{R}$? Or some other cardinality set?
- Is there a critical set with undefined density?
- What else is possible with partitions into critical sets? Does our example of an infinite square partitioned into three critical sets correspond somehow to a method for partitioning L_n into three critical sets for some values of n?

Paths to fame and fortune (Part 2)

- Find more critical sets in $L_{\mathbb{Z}}$, especially with densities other than $1/4$, $3/8$, $1/2$ and the values we already have up to $95/176$. In particular, what is the smallest density for a critical set of $L_{\mathbb{Z}}$.
- Same question, but for other infinite latin squares.
- What happens if we replace $\mathbb{Z}$ with $\mathbb{R}$? Or some other cardinality set?
- Is there a critical set with undefined density?
- What else is possible with partitions into critical sets? Does our example of an infinite square partitioned into three critical sets correspond somehow to a method for partitioning L_n into three critical sets for some values of n?

Disclaimer: To avoid the problems involved with infinite fame or fortune, those on offer are both *very* finite.

The End. Thank you!

The paper, which includes discussion and references to almost everything discussed here.

- A. Callahan, E. R. Hasson, K. Minde, M. A. Ollis and X. Zhu, Uniquely completable and critical subsets of the integer addition table, *Australasian Journal of Combinatorics* **89** (2024), 137–166.