Infinity, the axiom of choice, and mediacy

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they/she

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A question to the audience:

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What does it mean to be finite?

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What does it mean to be finite?

- We're talking about discrete collections, not continuous quantities.
- That is, we're talking about what a mathematician would call a set.

Some terminology first

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- If different inputs go to different outputs f is injective.
- If no output is missed f is surjective.
- If both f is bijective.

Many answers

X is infinite if. . .

-

X is finite if. . .

-

Many answers

X is infinite if...

- For every $n \in \mathbb{N}$ there's an injection $\{0, 1, \ldots, n-1\}$.
- There's an injective, non-surjective map $X \to X$.
- You can order X to have either no minimum or no maximum.
- There's a nonempty collection of subsets of X with no maximal element.
- X can be split into two pieces each of which has the same size as X.
- There is a bijection $X \to X \times X$.

X is finite if...

- There is bijective $\{0, 1, \ldots, n-1\} \to X$ for some $n \in \mathbb{N}$.
- Any injective $X \to X$ is bijective.
- Any linear order on X has both a minimum and a maximum.
- Any nonempty collection of subsets of X has a maximal element.
- If you split X into two pieces then both pieces are smaller than X.
- The square $X \times X$ is bigger than X.

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Are these answers all the same?

Dedekind's analysis

Richard Dedekind gave an analysis in *Was sind und was sollen die Zahlen?* (1888):

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Proposition: X is Dedekind-infinite if and only if there is injective $g : \mathbb{N} \to X$.

- ($\Leftarrow$) Push forward the $+1$ map to get an injective non-surjective $f : X \to X$.
- ($\Rightarrow$) Pick $z \in X \setminus \operatorname{ran} f$ to be zero, and 'plus one' is applying f.

Dedekind-infinite $\Rightarrow$ infinite

Dedekind-infinite $\Rightarrow$ infinite

- If X is Dedekind-infinite,
- then there is injective $g : \mathbb{N} \to X$,
- so restricting g to $\{0, 1, \ldots, n-1\}$ gives an injection,
- so X has at least n elements, for any $n \in \mathbb{N}$.

Infinite $\Rightarrow$ Dedekind-infinite.

This one's a little harder.

Infinite ⇒ Dedekind-infinite.

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How do we make these choices?

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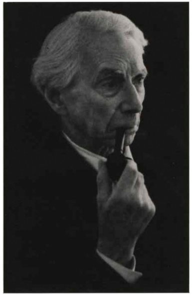

Bertrand Russell

"If you have infinitely many pairs of shoes you can pick a shoe from each pair by always grabbing the left, but if you have infinitely many pairs of socks how do you pick from each pair?"

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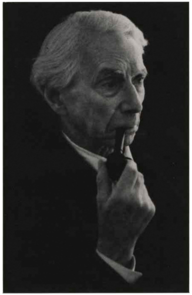

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Mathematically: if you have a rule to specify how to choose, you can always do that.

- If you have a bunch of sets of natural numbers, your rule can be to always pick the smallest number in each.
- If you have a bunch of open intervals (a, b), your rule can be to always pick the midpoint in each.

It's only when you don't have a rule that it's dicey.

- Challenge: Give a rule that tells you how to choose a number from *any* set of real numbers.

The axiom of choice, formally

- Let X be a set whose elements are (nonempty) sets.
- A choice function on X is a function c with domain X so that $c(x) \in x$ for every $x \in X$.
- The sets in X are the options for each of the choices, and c tells you what to choose for each.

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Often we can explicitly define a choice function, and so we don't need to use the axiom of choice to know it exists.

- If X consists of subsets x of $\mathbb{N}$, then $c(x) = \min x$ is a choice function for X.
- If X consists of open intervals (a, b) inside $\mathbb{R}$, then $c((a, b)) = (a + b)/2$ is a choice function for X.

The axiom of choice says we have a choice function even if we don't see how to define one.

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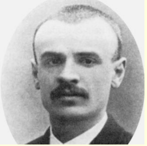
- The big critique was that the axiom of choice is non-constructive.



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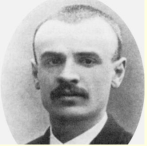

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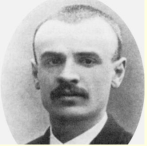
- Other mathematicians, such as Jacques Hadamard, defended Zermelo's axiom as a legitimate mathematical principle.
- And it turned out the analyst trio had used non-constructive principles in their own work.

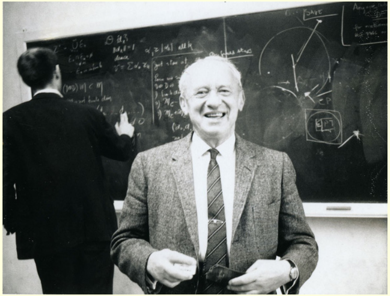


Many equivalences

The following are each equivalent to the axiom of choice:

- (Zermelo 1904) Zermelo's well-ordering theorem;
- (Hausdorff 1914) The Hausdorff maximal principle for partial orders;
- (Kuratowski 1922) Zorn's lemma;
- (Tarski 1924) If X is infinite there is a bijection $X \to X^2$;
- (Kelley 1950) Tychonoff's theorem on the product of topological spaces;
- (Hodges 1979) Krull's theorem about maximal ideals of rings;
- (Blass 1984) Every vector space has a basis.

Aside: A small case in the larger controversy



"Tarski told me the following story. He tried to publish his theorem [that the axiom of choice is equivalent to there being a bijection $X \to X^2$ for every infinite X] in the *Comptes Rendus Acad. Sci. Paris* but Fréchet and Lebesgue refused to present it. Fréchet wrote that an implication between two well known propositions is not a new result. Lebesgue wrote that an implication between two false propositions is of no interest. And Tarski said that after this misadventure he never tried to publish in the *Comptes Rendus*."

—Jan Mycielski, on p. 209 of "A system of axioms of set theory for the rationalists" (2006).

Back to infinite ⇒ Dedekind-infinite

This was our proof:

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Can we give rules for how to make those choices? If so, we know we didn't need the axiom of choice.

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The second choice:

- We need to pick an unused value in the range of f_{n+1}. Pick the one coming from the smallest $i < n+1$.

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Not seeing what to do isn't the same as proof of impossibility, however.

How could we possibly prove that something requires the axiom of choice?

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A parallel situation in geometry:

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Can you prove the axiom of choice as a theorem, assuming other basic axioms of set theory?

Some more history

- (1938) Kurt Gödel proved that the axiom of choice is consistent with the other axioms. You cannot prove it is false.
- His proof goes through a highly structured mathematical universe, the constructible universe, where you can define a global choice function—a choice function that works for every set simultaneously.

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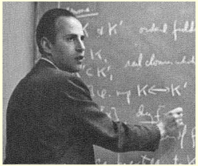

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- His proof goes through a highly structured mathematical universe, the constructible universe, where you can define a global choice function—a choice function that works for every set simultaneously.
- (1963) Paul Cohen proved that the failure of the axiom of choice is consistent with the other axioms. You cannot prove it is true.
- Altogether, the axiom of choice is independent of the other axioms.
- Cohen's proof introduced the method of forcing, a flexible technique for building new universes of mathematics.

Independence

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Later mathematicians built on these ideas to construct yet more universes where the axiom of choice fails in exciting ways.

- There are universes of mathematics where AC is false and basic analysis facts—e.g. the Baire category theorem, properties of the Borel sets, and properties of the Lebesgue integral—are also false.
- On the other hand, we know a weak fragment of the axiom of choice known as the principle of dependent choices is enough to prove these basic facts.
- For those who have heard of the Banach–Tarski paradox: in 1970 Robert Solovay constructed a universe where the principle of dependent choices is true and every set of reals is Lebesgue measurable.

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It follows from Cohen's work that there isn't a way to define it.

- If you could, then you would have the theorem without needing the axiom of choice.
- But Cohen proved it's consistent to have an infinite set which is Dedekind-finite.

Did we stumble upon another equivalence to the axiom of choice?
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Did we stumble upon another equivalence to the axiom of choice? Is "every infinite set is Dedekind-infinite" equivalent to AC?

The answer is no:

- One can check that the principle of dependent choices is enough to prove the implication.
- But set theorists know that the principle of dependent choices is weaker than the full axiom of choice: there are mathematical universes where DC is true but AC is false.
- So there are universes where the axiom of choice is false but every infinite set is Dedekind-infinite.

Intuition: This is a local fact, while the axiom of choice is a global principle.

A question

Question

Is there a suitable generalization of the notion of an infinite Dedekind-finite set whose nonexistence gives a characterization of AC?

Even more history

- What we've seen about infinite Dedekind-finite sets is, under other language, the state of the art for the first decade of AC's life.
- Mathematics is still awaiting Cohen to settle for certain that the axiom of choice is needed for the proof, but everyone conjectures it is.

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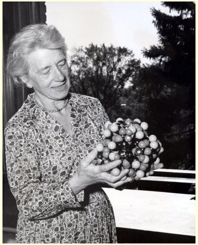
- Mathematics is still awaiting Cohen to settle for certain that the axiom of choice is needed for the proof, but everyone conjectures it is.
- In the late 1910s, Bertrand Russell is a few years after the last volume of his epic *Principia Mathematica*. His time is occupied by legal troubles over his pacifism during World War I and thinking about the foundations of mathematics.
- Working with him are multiple students, including Dorothy Wrinch.
- The next decade (1923) she will publish a paper answering our question.

Dorothy Wrinch



- Born 1894, died 1976.
- Studied logic under Russell, did her doctorate (1921) under applied mathematician John Nicholson.
- Wrote in a range of subjects: logic, pure mathematics, philosophy of science, and mathematical biology.
- Was awarded a Rockefeller Foundation fellowship to support her work in mathematical biology.
- Early career was in the UK, later emigrated to the USA. Latter years of her career were at Smith College (Mass, USA).
- Had the misfortune of being on the losing side of a scientific dispute with Linus Pauling over the structure of proteins.

Mediate sets

Some notation for sets X and Y:

- Write $X \leq Y$ if there is an injection $X \to Y$;
- Write $X \approx Y$ if there a bijection $X \to Y$;
- Write $X < Y$ if $X \leq Y$ but $X \not\approx Y$.

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Fix a set D. Then X is D-mediate if

- Whenever $Y < D$ then $Y \leq X$;
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This generalizes the notion of a set which is infinite but Dedekind-finite.

- "X is $\mathbb{N}$-mediate" is a rephrasing of "X is infinite and Dedekind-finite".
- The principle "every infinite set is Dedekind-infinite" can be rephrased "there are no $\mathbb{N}$-mediate sets".

A basic fact

Definition

Fix a set D. Then X is *D-mediate* if

- Whenever $Y < D$ then $Y \leq X$;
- But neither $D \leq X$ nor $X \leq D$

- You can prove—without the axiom of choice—that there are no finite degrees of mediacy.
- Use induction to prove that for every $n \in \mathbb{N}$ if there is no injection $X \to \{0, 1, \ldots, n-1\}$ then there is an injection $\{0, 1, \ldots, n-1\} \to X$.
- So the second clause of D-mediacy can never be true when D is finite.

The upshot: mediacy is only about infinite sets, the best kind of sets.

Wrinch's theorem

Theorem (Wrinch 1923)

Over the basic axioms of set theory, the following are equivalent.

1. AC; *and*
2. *There are no mediate sets.*

- Wrinch originally formulated this result in the framework of *Principia Mathematica*.
- Her same proof goes through in modern frameworks.

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- In particular, neither $X \leq D$ nor $D \leq X$.
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- In particular, neither $X \leq D$ nor $D \leq X$.
- So the principle of Cardinal Trichotomy fails.
- (Hartogs 1915) The axiom of choice is equivalent to Cardinal Trichotomy.
- So the axiom of choice fails.

Wrinch's theorem, ($2 \Rightarrow 1$)

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1. AC; *and*
2. *There are no mediate sets.*

Definition

Fix a set D. Then X is D-mediate if

- Whenever $Y < D$ then $Y \leq X$;
- But neither $D \leq X$ nor $X \leq D$

Prove ($2 \Rightarrow 1$) by contrapositive.

Wrinch's theorem, $(2 \Rightarrow 1)$

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Proof sketch:

- (Hartogs) For any set X there is a smallest ordinal $\aleph(X)$ so that there is no injection $\aleph(X) \to X$.
- If X is not well-orderable then X is $\aleph(X)$-mediate.
- Because the axiom of choice is equivalent to Zermelo's theorem that every set is well-orderable,
- If the axiom of choice fails there is a mediate set.

A second question

We know a lot more about the axiom of choice than was state of the art in Wrinch's day. Can we prove a better theorem?

Question

For which sets D is it consistent for there to be a D-mediate set?

We know $\mathbb{N}$ is possible. Any others?

Lévy's theorem

Theorem (Azriel Lévy (1964); independently W.)

Suppose there is a bijection from D to a regular ordinal. Then there is a universe in which there is a D-mediate set.

- An ordinal is morally a well-ordered set.
- An ordinal κ is regular if there is no cofinal map from a smaller ordinal to κ.

Lévy's theorem, stated more precisely

Theorem (Azriel Lévy (1964); independently W.)

Suppose $\kappa = \kappa^{<\kappa}$ *is regular. In the symmetric extension obtained from the forcing* $\mathrm{Add}(\kappa, \kappa)$ *by restricting to conditions which are hereditarily symmetric under permutations of the generics which fix* $< \kappa$ *many points, the following are true:*

- $\mathsf{DC}_{<\kappa}$;
- κ *is the smallest degree of mediacy.*

Those other notions of finiteness

Way back on slide 4 I gave a big list of different ways of defining finiteness (complementarily, infiniteness). We then ignored most of them to look only at Dedekind's.

- What about the others?

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Theorem (Lévy 1958)

Without assuming you the axiom of choice you cannot prove the equivalence of many various definitions of finiteness.

A question

Question

For Dedekind's characterization of finiteness, Wrinch gave a generalization which enabled an equivalence to the axiom of choice. Can a similar analysis be done for other characterizations?

Thank you!

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