Solid, neat tight: toward charting the boundary of definability

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Categoricity in second-order and first-order logic

Second-order logic allows quantifiers over subsets of the domain, not just elements.

- (Dedekind) ω is the unique model of Peano arithmetic, formulated in second-order logic.
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Suppose $\mathcal{M} \models \mathsf{PA}^2$. We build an isomorphism $\omega \cong \mathcal{M}$:

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First-order logic only allows quantifiers over elements. It cannot have such absolute categoricity results.

- (Löwenheim–Skolem) If a theory T has an infinite model then T has a model of every infinite cardinality $\geq |T|$.
- Trying to run Dedekind's construction for $\mathcal{M} \models \mathsf{PA}$ only gives that $\omega \hookrightarrow \mathcal{M}$ embeds as an initial segment.

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If something is impossible, as mathematicians we want to see how close we can get.

Question

Can we find categoricity-like properties which are enjoyed by the first-order logic formulations of important foundational theories like PA *or* ZF*?*

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To say what this means we need the notion of an interpretation.

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- An interpretation $\mathcal{I}$ of a structure $\mathcal{N}$ in $\mathcal{M}$ is a collection of formulae which gives an isomorphic copy of $\mathcal{N}$ in $\mathcal{M}$: one formula for the domain, others for the functions and relations.
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- $\mathsf{ACA}_0 \trianglerighteq \mathsf{PA}$ but $\mathsf{PA} \not\trianglerighteq \mathsf{ACA}_0$
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Fact: Doing $\mathsf{ZFC}^{\neg\infty} \trianglerighteq \mathsf{PA}$ then $\mathsf{PA} \trianglerighteq \mathsf{ZFC}^{\neg\infty}$ or vice versa gives an isomorphism.

But that's not true for doing $\mathsf{ZF} \trianglerighteq \mathsf{ZFC} + \mathrm{V} = \mathrm{L}$ then $\mathsf{ZFC} + \mathrm{V} = \mathrm{L} \trianglerighteq \mathsf{ZF}$.

Bi-interpretations

U is a retract of T if

- $U \trianglerighteq^{\mathcal{I}} T \trianglerighteq^{\mathcal{J}} U$ and $\mathcal{J} \circ \mathcal{I}$ is definably isomorphic to the identity interpretation on U.

- $\mathcal{M} \trianglerighteq^{\mathcal{I}} \mathcal{N} \trianglerighteq^{\mathcal{J}} \mathcal{M}^* \Longrightarrow \mathcal{M} \cong^{\mathcal{J} \circ \mathcal{I}} \mathcal{M}^*$

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- $\mathrm{ZFC} + \mathrm{V} = \mathrm{L}$ is a retract of ZF.
- But ZF and $\mathrm{ZFC} + \mathrm{V} = \mathrm{L}$ are not bi-interpretable (Enayat).

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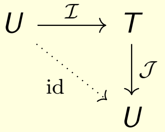

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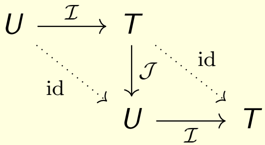

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But we avoid loops:

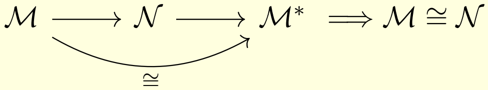
- If $\omega \trianglerighteq_{\text{par}} \mathcal{N} \trianglerighteq_{\text{par}} \omega$ then $\mathcal{N} \cong \omega$.
(Because a model of arithmetic cannot interpret a shorter model.)

Solidity

A theory T is solid if

- For all models $\mathcal{M}, \mathcal{M}^*, \mathcal{N}$ of T if
  - $\mathcal{M} \trianglerighteq_{\text{par}} \mathcal{N} \trianglerighteq_{\text{par}} \mathcal{M}^*$ and
  - There is a parametrically definable isomorphism $\mathcal{M} \cong \mathcal{M}^*$,

  Then there is a parametrically definable isomorphism $\mathcal{M} \cong \mathcal{N}$.



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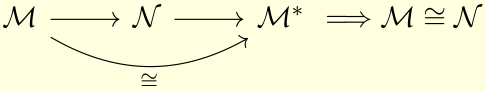
Example:

- (Visser) PA is solid.

Because the "$\omega \trianglerighteq \mathcal{N} \trianglerighteq \omega$ implies $\mathcal{N} \cong \omega$" argument can be made to work over any $\mathcal{M} \models \text{PA}$.



Visser's theorem in more detail

Theorem (Visser)

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Lemma: If $\mathcal{M} \trianglerighteq_{\text{par}} \mathcal{N}$ are models of PA then $\mathcal{M}$ embeds as an initial segment of $\mathcal{N}$.

- Exactly like the argument that ω is an initial segment of any model of PA.

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Proof Sketch: Suppose $i : \mathcal{M} \to \mathcal{N}$ and $j : \mathcal{N} \to \mathcal{M}^*$ are definable initial embeddings and $\mathcal{M} \cong_{\text{par}} \mathcal{M}^*$.

- Claim: i and j are both surjective.
  - Else, $k = j \circ i : \mathcal{M} \to \mathcal{M}^*$ embeds $\mathcal{M}$ onto a strict initial segment of $\mathcal{M}^*$.
  - But composing k with the isomorphism $\mathcal{M}^* \cong \mathcal{M}$ gives a definable cut in $\mathcal{M}$.
  - This is impossible, since in PA any bounded definable set has a maximum.
- So i is the desired isomorphism. □

Neatness and tightness

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- But the converses do not hold.
- All of these properties are preserved by bi-interpretations.
- All of these properties are preserved by adding axioms (in the same language).
- These properties are really only interesting for sequential theories—those which are subject to the first incompleteness theorem.
- A complete theory such as ACF_0 is trivially neat.

Positive examples

Theorem

The following theories are all solid, and hence also neat and tight.

- *(Visser)* PA
- *(Enayat)* ZF
- *(Enayat)* Z_2, *second-order arithmetic with full comprehension*
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Recent work (Piotr Gruza, Leszek Kołodziejczyk, and Mateusz Łełyk): No. There are theories intermediate between $I\Sigma_k$ and PA which are solid.

Negative examples

Theorem

None of the following are tight, and hence are neither neat nor solid.

- *(Freire–Hamkins) Zermelo set theory*
- *(Freire–Hamkins)* ZF^-, *set theory without Powerset*
- *(Enayat) Finite subtheories of* PA, ZF, Z_2, *or* KM
- *(Freire–W.)* ACA *and* Π^1_k-CA, *i.e. with full induction but only bounded comprehension, and the analogous subtheories of* KM

These results suggest that tightness characterizes the important foundational theories like PA and ZF.

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Theorem

ACA *is not tight: there are distinct but bi-interpretable extensions of* ACA.

- ACA is the subsystem of second-order arithmetic whose primary axioms are arithmetical comprehension and full induction.
- Any ω-model of ACA_0 is a model of ACA.

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- For each $k \in \omega$, the k-th jump $0^{(k)}$ is arithmetical.
- So we can define $0^{(\omega)}$ by identifying which sets are the $0^{(k)}$ then gluing them together.
- Key point: The $0^{(k)}$ are not *uniformly* arithmetical, but the property of being a $0^{(k)}$ is uniformly recognizable.

Identifying the arithmetical sets

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- Indeed, it has a definition absolute between all ω-models of ACA ($=$ Turing ideals closed under jump $= \mathcal{X} \subseteq \mathcal{P}(\omega)$ closed under arithmetical comprehension).
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- We just saw a Σ^1_1 definition. There's also a Π^1_1 definition.

Forcing over the arithmetical sets

- We can add a new real by finite approximations.
- $\mathbb{C}$ is the poset consisting of finite partial functions $\omega \to 2$, ordered by extension.
- A real $c \subseteq \omega$ is generic over a Turing ideal $\mathcal{X}$ if it get below every dense set in $\mathcal{X}$. ($D \subseteq \mathbb{C}$ is dense if any $p \in \mathbb{C}$ extends to an element of D.)
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- Since $0^{(\omega)}$ is Δ^1_1-definable over any ω-model of ACA we get that any ω-model of ACA can define a generic over the arithmetical sets.
- Indeed, they can all define the same generic, call it $\mathfrak{c}$.

Key point: From $0^{(\omega)}$ you can extract a canonical enumeration of the arithmetical sets, and you use that enumeration to construct $\mathfrak{c}$.

Bi-interpretations

Two structures:

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- ACA has full induction, which makes the arguments about defining $0^{(\omega)}$ and $\mathfrak{c}$ work, even over an ω-nonstandard model.
- The definitions are sufficiently absolute to enable a bi-interpretation:
  - ACA + "I am the arithmetical sets" and
  - ACA + "I am the sets arithmetical in $\mathfrak{c}$".

Thus, ACA is not tight. □

Generalizing the non-tightness construction

Abstractly, these are the ingredients we need:

- A canonical structure;
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For ACA:

- The arithmetical sets;
- Cohen forcing;
- The absoluteness of $0^{(\omega)}$;
- Given by the induction schema.

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Can be done for Π^1_k-CA:

- The minimum β-model of Π^1_k-CA;
- Cohen forcing;
- The absoluteness of L;
- A little fine structure theory.

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For class theories $T \subseteq \mathsf{KM}$:

- Minimum models again;
- Cohen forcing again;
- L again;
- Fine structure theory again.

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Other uses?

- Maybe only need the first three?
- Or just two of them?

Back to Enayat's conjecture

Conjecture (Enayat)

A theory T of arithmetic is tight if and only if $T \supseteq \mathsf{PA}$.
And similarly for ZF and other important foundational theories.

While we now know the general conjecture to be false (Gruza, Kołodziejczyk, and Łełyk), many natural fragments of PA, etc. fail to be tight.

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Constructions for other negative results have a similar flavor.

A moral: These categoricity-like properties are characterizing semantic closure—the limits of definability.

Some open questions

- Is there a finitely axiomatizable sequential tight theory?
  (Enayat) No for subtheories of PA and ZF.
- Is $\mathrm{PA}^- + \text{Collection}$ tight?
  (Enayat−Łełyk) It is not solid.
- Is there an extension of KP which is solid?
- Can we better understand the separation between solidity, neatness, and tightness?
  - Recent work (Piotr Gruza, Leszek Kołodziejczyk, and Mateusz Łełyk): There are theories intermediate between $\mathrm{I}\Sigma_n$ and PA which are neat but not tight.

Thank you!

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