Solid, neat tight: toward charting the boundary of definability

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Categoricity in second-order logic

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First-order logic only allows quantifiers over elements. It cannot have such absolute categoricity results.

- (Löwenheim–Skolem) If a theory T has an infinite model then T has a model of every infinite cardinality $\geq |T|$.

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If something is impossible, as mathematicians we want to see how close we can get.

Question

Can we find categoricity-like properties which are enjoyed by the first-order logic formulations of important foundational theories like PA *or* ZF*?*

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To say what this means we need the notion of an interpretation.

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- An interpretation $\mathcal{I}$ of a structure $\mathcal{N}$ in $\mathcal{M}$ is a collection of formulae which gives an isomorphic copy of $\mathcal{N}$ in $\mathcal{M}$: one formula for the domain, others for the functions and relations.
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- $\mathsf{ZFC} + \mathrm{V} = \mathrm{L} \trianglerighteq \mathsf{ZF}$
- $\mathsf{ACA}_0 \trianglerighteq \mathsf{PA}$ but $\mathsf{PA} \ntrianglerighteq \mathsf{ACA}_0$
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Fact: Doing $\mathsf{ZFC}^{\neg\infty} \trianglerighteq \mathsf{PA}$ then $\mathsf{PA} \trianglerighteq \mathsf{ZFC}^{\neg\infty}$ or vice versa gives an isomorphism.

But that's not true for doing $\mathsf{ZF} \trianglerighteq \mathsf{ZFC} + \mathrm{V} = \mathrm{L}$ then $\mathsf{ZFC} + \mathrm{V} = \mathrm{L} \trianglerighteq \mathsf{ZF}$.

Bi-interpretations

U is a retract of T if

- $U \trianglerighteq^{\mathcal{I}} T \trianglerighteq^{\mathcal{J}} U$ and $\mathcal{J} \circ \mathcal{I}$ is definably isomorphic to the identity interpretation on U.

- $\mathcal{M} \trianglerighteq^{\mathcal{I}} \mathcal{N} \trianglerighteq^{\mathcal{J}} \mathcal{M}^* \implies \mathcal{M} \cong^{\mathcal{J} \circ \mathcal{I}} \mathcal{M}^*$

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- $\mathsf{ZFC} + \mathrm{V} = \mathrm{L}$ is a retract of ZF.
- But ZF and $\mathsf{ZFC} + \mathrm{V} = \mathrm{L}$ are not bi-interpretable (Enayat).

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But we avoid loops:

- If $\omega \trianglerighteq_{\text{par}} \mathcal{N} \trianglerighteq_{\text{par}} \omega$ then $\mathcal{N} \cong \omega$.
  (Because a model of arithmetic cannot interpret a shorter model.)

Solidity

A theory T is solid if

- For all models $\mathcal{M}, \mathcal{M}^*, \mathcal{N}$ of T
  - If $\mathcal{M} \trianglerighteq_{\text{par}} \mathcal{N} \trianglerighteq_{\text{par}} \mathcal{M}^*$ and
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Example:

- (Visser) PA is solid.

Because the "$\omega \trianglerighteq \mathcal{N} \trianglerighteq \omega$ implies $\mathcal{N} \cong \omega$" argument can be made to work over any $\mathcal{M} \models \mathrm{PA}$.

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- Solidity implies neatness and neatness implies tightness.
- But the converses do not hold.
- All of these properties are preserved by bi-interpretations.
- All of these properties are preserved by adding axioms (in the same language).
- These properties are really only interesting for sequential theories—those which are subject to the first incompleteness theorem.
- A complete theory such as ACF_0 is trivially neat.

Positive examples

Theorem

The following theories are all solid, and hence also neat and tight.

- *(Visser)* PA
- *(Enayat)* ZF
- *(Enayat)* Z_2, *second-order arithmetic with full comprehension*
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Question (Enayat): Do we need the full strength of these theories to get these quasi-categoricity properties?

Negative examples

Theorem

None of the following are tight, and hence are neither neat nor solid.

- *(Freire–Hamkins) Zermelo set theory*
- *(Freire–Hamkins)* ZF^-, *set theory without Powerset*
- *(Enayat) Finite subtheories of* PA, ZF, Z_2, *or* KM
- *(Freire–W.)* ACA *and* Π^1_k-CA, *i.e. with full induction, and the analogous subtheories of* KM

These results suggest that tightness characterizes the important foundational theories like PA and ZF.

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ACA *is not tight: there are distinct but bi-interpretable extensions of* ACA.

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- So we can define $0^{(\omega)}$ by identifying which sets are the $0^{(k)}$ then gluing them together.
- Key point: The $0^{(k)}$ are not *uniformly* arithmetical, but the property of being a $0^{(k)}$ is uniformly recognizable.

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- Indeed, it has a definition absolute between all ω-models of ACA (= Turing ideals closed under jump = $\mathcal{X} \subseteq \mathcal{P}(\omega)$ closed under arithmetical comprehension).
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- We just saw a Σ^1_1 definition. There's also a Π^1_1 definition.

Forcing over the arithmetical sets

- We can add a new real by finite approximations.
- $\mathbb{C}$ is the poset consisting of finite partial functions $\omega \to 2$, ordered by extension.
- A real $c \subseteq \omega$ is generic over a Turing ideal $\mathcal{X}$ if it get below every dense set in $\mathcal{X}$.
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- Since $0^{(\omega)}$ is Δ^1_1-definable over any ω-model of ACA we get that any ω-model of ACA can define a generic over the arithmetical sets.
- Indeed, they can all define the same generic, call it c.

Bi-interpretations

Two structures:

$$(\omega, \mathcal{A}) \qquad \text{and} \qquad (\omega, \mathcal{A}[c])$$

$\mathcal{A}$ is the arithmetical sets.

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- The two structures interpret each other.
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- ACA has full induction, which makes the arguments about defining $0^{(\omega)}$ and c work, even over an ω-nonstandard model.
- The definitions are sufficiently absolute to enable a bi-interpretation:
  - ACA + "I am the arithmetical sets" and
  - ACA + "I am the sets arithmetical in c".

Thus, ACA is not tight.

Generalizing the construction

Abstractly, these are the ingredients we need:

- A canonical structure;
- How to extend this structure;
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For ACA:

- The arithmetical sets;
- Cohen forcing;
- The absoluteness of $0^{(\omega)}$;
- Given by the induction schema.

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Can be done for Π^1_k-CA:

- The minimum β-model of Π^1_k-CA;
- Cohen forcing;
- The absoluteness of L;
- A little fine structure theory.

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For class theories $T \subseteq \mathsf{KM}$:

- Minimum models again;
- Cohen forcing again;
- L again;
- Fine structure theory again.

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Other uses?

- Maybe only need the first three?
- Or just two of them?

Back to Enayat's conjecture

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- The arithmetical sets lack semantic closure.
  Over them you can define sets which are not arithmetical.

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A moral: These categoricity-like properties are characterizing semantic closure.

Some open questions

- Is there a finitely axiomatizable sequential tight theory?
  (Enayat) No for subtheories of PA and ZF.
- Is $PA^- +$ Collection tight?
  (Enayat–Łełyk) It is not solid.
- Is there an extension of KP which is solid?

Thank you!

References

- Ali Enayat, “Variations on a Visserian theme”, In: *Liber Amicorum Alberti, a Tribute to Albert Visser* (2016).
  Preprint: arXiv:1702.07093 [math.LO].
- Ali Enayat and Mateusz Łełyk, “Categoricity-like properties in the first-order realm”, *under review*.
  Preprint: `https://www.researchgate.net/publication/377931753`
- Alfredo Roque Freire and Kameryn J. Williams, “Non-tightness in class theory and second-order arithmetic”, JSL (2023).
  Preprint: arXiv:2212.04445 [math.LO].
- Albert Visser, “Categories of theories and interpretations”, In: *Logic in Tehran* (2006).