A nonstandard approach to integer combinatorics

Kameryn J. Williams
they/them

Bard College at Simon's Rock

Boise State TATERS
2023 Nov 17

Joint work with Timothy Trujillo (Sam Houston State University)

A nuanced and detailed history of the calculus

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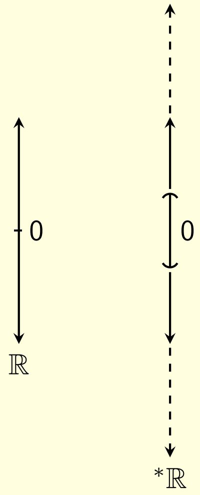
"Leibniz was right all along. Using model theory we can make infinitesimal calculus mathematically rigorous."

Nonstandard analysis in a nutshell

Use the model-theoretic notion of an ultrapower to embed $\mathbb{R}$ into a saturated elementary extension ${}^*\mathbb{R}$.

- Any standard object f on $\mathbb{R}$ has a nonstandard extension ${}^*f$ with the same elementary properties.
- You can transfer properties in ${}^*\mathbb{R}$ back to $\mathbb{R}$.



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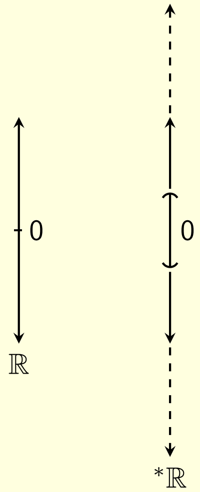
The first big new result using NSA was:

- (Bernstein & Robinson 1966) Any polynomially compact operator on a Hilbert space has an invariant subspace.



It's not just for analysts

- Looking at an embedding $\mathfrak{A} \hookrightarrow {}^*\mathfrak{A}$ can be done for any mathematical structure $\mathfrak{A}$.
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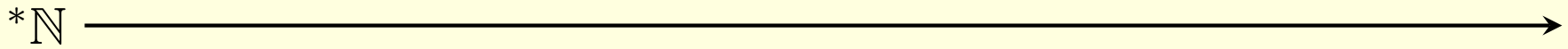
This is the application of nonstandard methods we'll care about for the rest of the hour.

What does ${}^*\mathbb{N}$ even look like?

${}^*\mathbb{N}$ is a discretely ordered semiring.

Elementarity: Any property of $\mathbb{N}$ expressed just by quantifying over numbers is true in ${}^*\mathbb{N}$.



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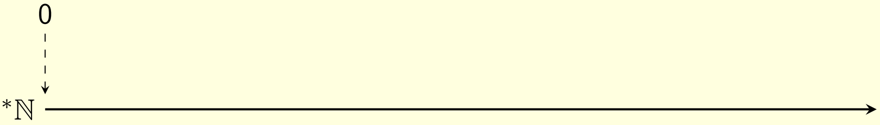

$^*\mathbb{N}$ has the additive identity 0 as its least element, because $\forall n\ 0 \leq n$ is true in $\mathbb{N}$.

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"$\forall n \; n < 3$ iff $n = 0$ or $n = 1$ or $n = 1 + 1$" is true in $\mathbb{N}$

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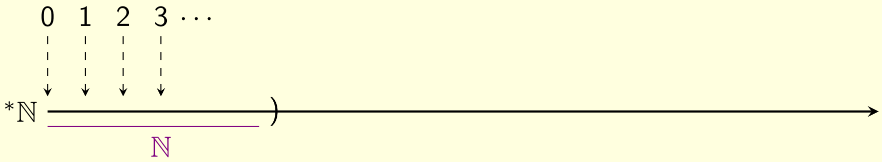

$\mathbb{N}$ embeds as an initial segment into $^*\mathbb{N}$. The new elements are all hyperfinite.

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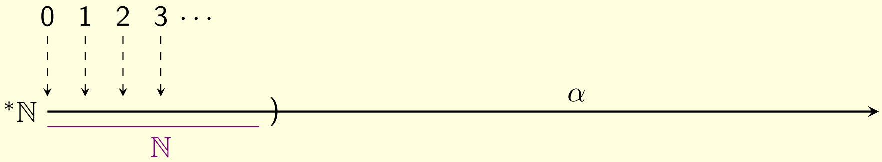

If $\alpha \in {}^*\mathbb{N} \setminus \mathbb{N}$ then $\alpha > n$ for all $n \in \mathbb{N}$.

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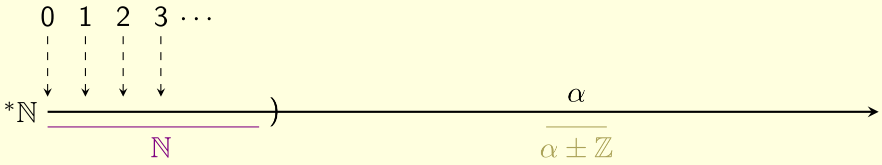

All non-zero elements have a predecessor

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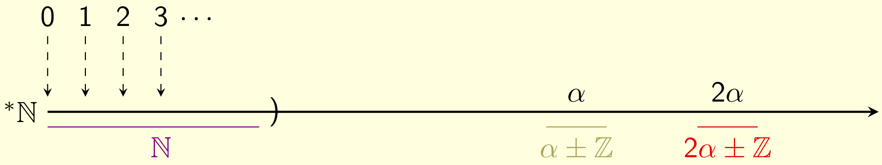

$\alpha + n < \alpha + \alpha = 2\alpha$ for all $n \in \mathbb{N}$.

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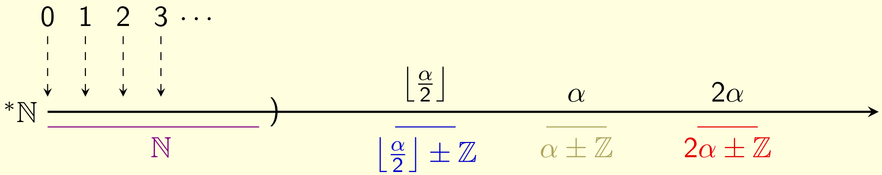

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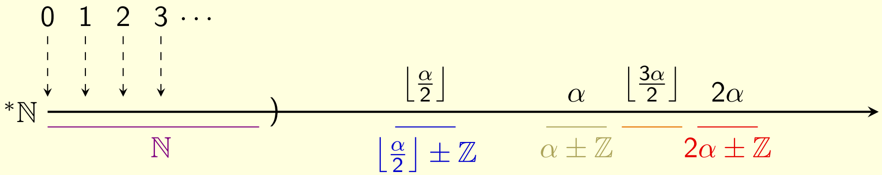

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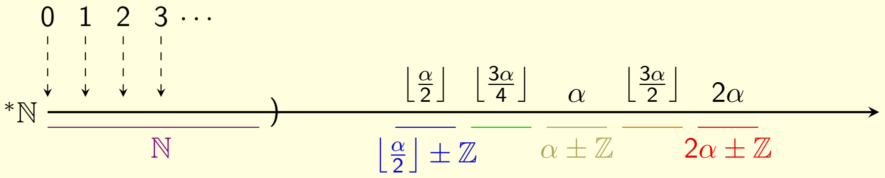

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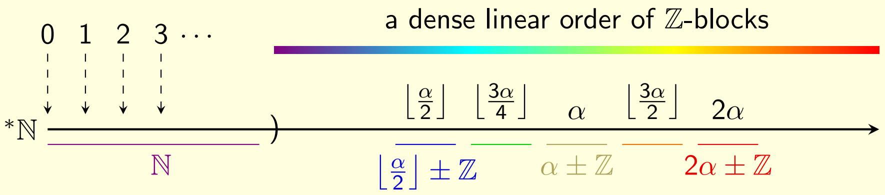

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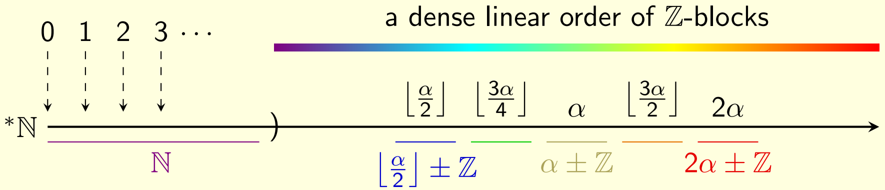
Saturation: If a sequence of elementary properties $\varphi_0(x), \varphi_1(x), \ldots$ is finitely consistent in $\mathbb{N}$, then you can find nonstandard α so all $\varphi_n(\alpha)$ hold simultaneously.



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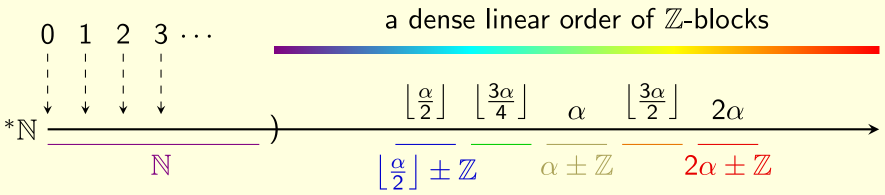

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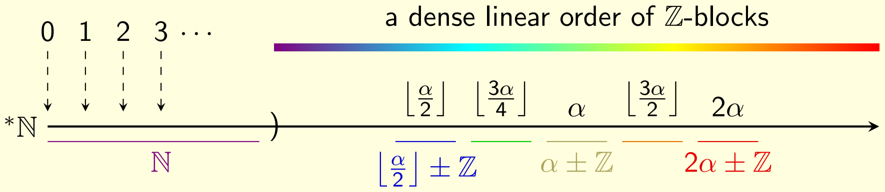

If $P \subseteq \mathbb{N}$ is your favorite set of primes, there's nonstandard α so that $p \mid \alpha$ iff $p \in P$.
Therefore ${}^*\mathbb{N}$ is uncountable.

Some important transfer properties

- Elementarity: Any property of $\mathbb{N}$ you can express just by quantifying over numbers is true in $^*\mathbb{N}$.
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Useful special cases of elementarity:

- Preservation of partitions:
  If $\Pi = \{X_0, \ldots, X_n\}$ is a finite partition of $\mathbb{N}$, then ${}^*\Pi = \{{}^*X_0, \ldots, {}^*X_n\}$ is a finite partition of ${}^*\mathbb{N}$.
- Characterization of infinite:
  $X \subseteq \mathbb{N}$ is infinite iff there is some nonstandard $\alpha \in {}^*X$.
- Preservation of finiteness:
  If X is finite then so is ${}^*X = \{{}^*x : x \in X\}$.

Enough preliminaries, let's take this for a drive

The pigeonhole principle

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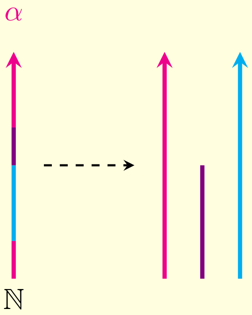
Proof:

- Consider $\alpha \in {}^*\mathbb{N} \setminus \mathbb{N}$.
- ${}^*X_0, \ldots, {}^*X_n$ are a partition of ${}^*\mathbb{N}$.
- So α is in some ${}^*X_i$.
- So X_i is infinite.



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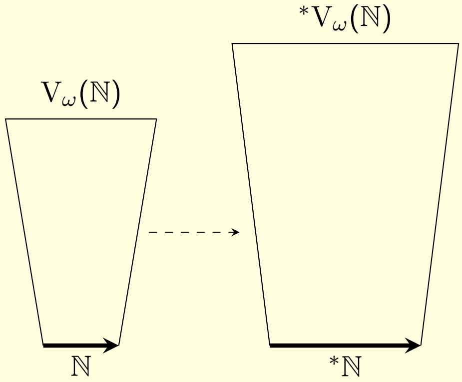
- Actually we embed $V_\omega(\mathbb{N})$ into a saturated elementary extension.
- $V_\omega(\mathbb{N}) = \mathbb{N} \cup \mathcal{P}(\mathbb{N}) \cup \mathcal{P}(\mathcal{P}(\mathbb{N})) \cup \cdots$
- The ultrafilter used to construct the extension is an element of $V_\omega(\mathbb{N})$.
- So $^*V_\omega(\mathbb{N})$ is a subset of $V_\omega(\mathbb{N})$.
- So $^*\mathbb{N}$ is in the domain of the embedding.
- We can apply the * map to $^*\mathbb{N}$ itself.
- If $\alpha \in {^*\mathbb{N}} \setminus \mathbb{N}$ then $\alpha < {^*\alpha}$.
- And we can iterate:

$$\mathbb{N} \hookrightarrow {^*\mathbb{N}} \hookrightarrow {^{*(2)}\mathbb{N}} \hookrightarrow \ldots \hookrightarrow {^{*(k)}\mathbb{N}} \hookrightarrow \ldots$$



Ramsey's theorem

Theorem (Ramsey 1930)

Partition $[\mathbb{N}]^k$ *into finitely many pieces* $X_0, \ldots, X_n$. *Then there is infinite* $H \subseteq \mathbb{N}$ *so that* $[H]^k \subseteq X_i$ *for some* i.

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Proof ($k = 3$):

- Consider $\alpha \in {}^*\mathbb{N} \setminus \mathbb{N}$.
- Then $\langle \alpha, {}^*\alpha, {}^{*(2)}\alpha \rangle$ is in some ${}^{*(3)}X_i$.
- $A_\emptyset = \{a \in \mathbb{N} : \langle a, \alpha, {}^*\alpha \rangle \in {}^{*(2)}X_i\}$.
- ${}^*A_\emptyset = \{a \in {}^*\mathbb{N} : \langle a, {}^*\alpha, {}^{*(2)}\alpha \rangle \in {}^{*(3)}X_i\}$.
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Do an induction:

- Already built $H_i = \langle h_0, \ldots, h_i \rangle$.
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- $\alpha \in \bigcap_{t \in [H_i]^{<3}} {}^*A_t = {}^*\left(\bigcap_{t \in [H_i]^{<3}} A_t \right)$.
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- Pick $h_{i+1} > h_i$ from that intersection.

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Finally $H = \langle h_i \rangle$ is monochromatic.

Compare to standard proofs of Ramsey's theorem

- Also goes by induction. At stage i, have built up H_i an initial segment of the monochromatic H.
- For $t \in [H_i]^{<3}$, have A_t is the set of ways you can extend t to get a tuple of the correct color.

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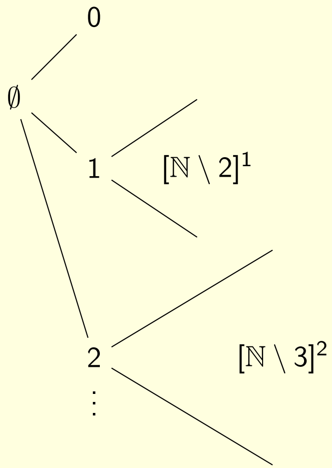
"I do not think that a scientific result which gives us a better understanding of the world and makes it more harmonious in our eyes should be held in lower esteem than an invention which improves household plumbing." –Alfred Tarski (paraphrased)

Generalizing Ramsey to families of sets of nonuniform size

Definition

The Schreier barrier $\mathcal{S}$ consists of all $s \in [\mathbb{N}]^{<\omega}$ so that $|s| = \min s + 1$.

- The first element of s tells you how long s is.
- You can think of $\mathcal{S}$ as a tagged amalgamation of (copies of) all $[\mathbb{N}]^k$.



A Ramsey property for the Schreier barrier

Theorem (Nash-Williams for $\mathcal{S}$)

Partition $\mathcal{S}$ into finitely many pieces. Then there is infinite $H \subseteq \mathbb{N}$ so that $\mathcal{S} \restriction H$ is monochromatic.

$$\mathcal{S} \restriction H = \{s \in \mathcal{S} : s \subseteq H\}$$

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- For $[\mathbb{N}]^k$ we looked at what piece of the partition contained $\langle \alpha, {}^*\alpha, \ldots, {}^{*(k-1)}\alpha \rangle$
- But now we don't know in advance how long a sequence in $\mathcal{S}$ will be
- Intuitively, we want to look at

$$\langle \alpha, {}^*\alpha, \ldots {}^{*(\alpha)}\alpha \rangle$$

- But this is nonsensical—what would it even mean to iterate * a hyperfinite number of times?

A proxy for $\langle \alpha, {}^*\alpha, \ldots {}^{*(\alpha)}\alpha \rangle$

Fact: Fix $\alpha \in {}^*\mathbb{N}$. There is (a non-unique) $\sigma(\alpha)$ so that for any set X

$$\sigma(\alpha) \in {}^*X \quad \Leftrightarrow \quad \alpha \in {}^*\{k \in \mathbb{N} : \langle \alpha, \ldots, {}^{*(k-1)}\alpha \rangle \in {}^{*(k)}X\}.$$

This $\sigma(\alpha)$ is a proxy for $\langle \alpha, {}^*\alpha, \ldots, {}^{*(\alpha)}\alpha \rangle$.

A proxy for $\langle \alpha, {}^*\alpha, \ldots {}^{*(\alpha)}\alpha \rangle$

Fact: Fix $\alpha \in {}^*\mathbb{N}$. There is (a non-unique) $\sigma(\alpha)$ so that for any set X

$$\sigma(\alpha) \in {}^*X \quad \Leftrightarrow \quad \alpha \in {}^*\{k \in \mathbb{N} : \langle \alpha, \ldots, {}^{*(k-1)}\alpha \rangle \in {}^{*(k)}X\}.$$

This $\sigma(\alpha)$ is a proxy for $\langle \alpha, {}^*\alpha, \ldots, {}^{*(\alpha)}\alpha \rangle$.

- Just like $\langle \alpha, {}^*\alpha, {}^{*(2)}\alpha \rangle$ was used to guide our choices to construct a monochromatic set for $[\mathbb{N}]^3$,
- Use $\sigma(\alpha)$ to guide the choices to build a monochromatic set for the Schreier barrier.

Further generalization: fronts

$\mathcal{F} \subseteq [\mathbb{N}]^{<\omega}$ is a front if

- (antichain or Nash-Williams property) distinct elements of $\mathcal{F}$ cannot be initial segments of each other
- (density) any infinite $b \subseteq \mathbb{N}$ has an initial segment in $\mathcal{F}$

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Examples:

- $[\mathbb{N}]^k$ for any k
- The Schreier barrier $\mathcal{S}$

To prove a Ramsey property for $[\mathbb{N}]^k$ and $\mathcal{S}$ we had an idea of what a generic nonstandard member looked like, based on how the front was built up.

- $\langle \alpha, \ldots, {}^{*(k-1)}\alpha \rangle$ for $[\mathbb{N}]^k$
- $\sigma(\alpha)$, a proxy for $\langle \alpha, \ldots, {}^{*(\alpha)}\alpha \rangle$ for $\mathcal{S}$

If we want to do the same for an arbitrary front $\mathcal{F}$ we need to understand how $\mathcal{F}$ is built up.

The Nash-Williams theorem for Ellentuck space

Theorem (Nash-Williams theorem)

Let $\mathcal{F}$ be a front. Partition $\mathcal{F}$ into finitely many pieces. Then there is infinite $H \subseteq \mathbb{N}$ so that $\mathcal{F} \restriction H$ is monochromatic.

$\mathcal{F} \restriction H = \{s \in \mathcal{F} : s \subseteq H\}$

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- Fronts can be understood as inductively built up from simpler fronts.
- Inductively along the tree of subfronts of $\mathcal{F}$ you can build up a hyperobject $\sigma_{\mathcal{F}}(\alpha)$.
- Use $\sigma_{\mathcal{F}}(\alpha)$ to guide the choices to build a monochromatic set for $\mathcal{F}$.

The Point: Do the same proof as for Ramsey's theorem, but with a fancier object to guide the induction.

The topological in topological Ramsey theory

It was realized that a lot of combinatorial theorems about $\mathbb{N}$ could be understood as expressing different facets of a certain topological space.

Ellentuck space $\mathcal{E}$ has multiple components.

- The points are infinite subsets of $\mathbb{N}$.
- You can associate to each point its k-th finite approximation in $[\mathbb{N}]^k$.
- There is a partial order $\subseteq$ on points.

The Ellentuck topology on $\mathcal{E}$ is generated by basic open sets

$$[t, X] = \{Y \in \mathcal{E} : Y \subseteq X \text{ and } t \sqsubseteq Y\}.$$

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Get a connection between topology and combinatorics:

- $\mathcal{X} \subseteq \mathcal{E}$ is Ramsey if you can refine any basic open set be either contained in or disjoint from $\mathcal{X}$.
- $\mathcal{X} \subseteq \mathcal{E}$ is Ramsey null if it is Ramsey and you can always refine to be disjoint from $\mathcal{X}$.
- **Fact:** Any Baire subset of $\mathcal{E}$ is Ramsey and any meager subset is Ramsey null.
- Indeed any Souslin-measurable or Borel subset is Ramsey.

Abstract Ramsey spaces

Ellentuck space $\mathcal{E}$ has some nice properties.

(A.1) Sequencing: points behave like infinite sequences.

(A.2) Finitization: you can port the partial order $\subseteq$ to the finite approximations, and each approximation has a finite number of predecessors.

(A.3) Amalgamation: [this one's more technical].

(A.4) Pigeonhole: as it says in the name.

A Ramsey space is a tuple $(\mathcal{R}, \mathcal{AR}, \leq, r)$ satisfying (A.1–4) where $\mathcal{R}$ are the points, $r : \mathcal{R} \times \mathbb{N} \to \mathcal{AR}$ is the finite approximation map, and $\leq$ is the partial order.

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- You can put an Ellentuck topology on $\mathcal{R}$, and get a topology $\Leftrightarrow$ combinatorics connection.

The abstract Nash-Williams theorem

Theorem (Abstract Nash-Williams)

Suppose $\mathcal{R}$ is closed (in the product topology on $\mathcal{AR}$). Then any front on the finite approximations $\mathcal{AR}$ satisfies a Ramsey partition property.

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The abstract Nash-Williams theorem

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- I'd like to say our nonstandard proof of the Nash-Williams theorem extends directly to the full abstract Nash-Williams theorem.
- But we need the space to be amenable to nonstandard methods.
- And we don't (yet?) have a proof that this applies to every nontrivial Ramsey space.

What we do have for the abstract Nash-Williams theorem

Under an extra assumption the nonstandard proof goes through.

Theorem (Partial abstract Nash-Williams)

Consider a front $\mathcal{F}$ on $\mathcal{AR}$. Suppose

- *$\mathcal{AR}$ is infinitely branching everywhere; and*
- *There is a filter $\mathcal{C}$ on $\mathcal{R}$ so that for each $s \in T(\mathcal{F}) \setminus \mathcal{F}$ the restriction of $\operatorname{succ} s$ to $\mathcal{C}$ is a nonprincipal ultrafilter on $\operatorname{succ} s$.*

Then $\mathcal{F}$ satisfies a Ramsey partition property.

- $(\mathcal{R}, \leq)$ is a poset, so the usual definition of filter applies to $\mathcal{C}$
- $\operatorname{succ} s \restriction X = \{t \in \operatorname{succ} s : \exists k \; t \leq_{\mathrm{fin}} r_k(X)\}$
- $\operatorname{succ} s \restriction \mathcal{C} = \{\operatorname{succ} s \restriction X : X \in \mathcal{C}\}$

Positive examples

Any Ramsey space which can be thought of as its $(k+1)$-th approximations coming from k-th approximations by concatenating sequences from (cofinite subsets of) a countable alphabet will satisfy the extra assumption we need.

- Ellentuck space
- The Milliken space of block sequences
- The Hales–Jewett space of variable words
- The space $\mathcal{E}_\omega(\mathbb{N})$ of equivalence relations on $\mathbb{N}$ with infinite quotients

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What else?

Continuing work

- The abstract Nash-Williams theorem isn't the only theorem in abstract Ramsey theory.
- What other results are amenable to nonstandard methods?

Thank you!