Non-tightness in class theory

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they/them

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Joint work with Alfredo Roque Freire

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- ZF isn't finitely axiomatizable;
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- (Reflection) For each finite set T of axioms from ZF, ZF proves there is a club of ordinals α so that $\mathrm{V}_\alpha \models T$.

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- (Reflection) For each finite set T of axioms from ZF, ZF proves there is a club of ordinals α so that $\mathrm{V}_\alpha \models T$.
- If T_0, T_1 are extensions of ZF, then T_0 and T_1 are bi-interpretable iff they have the same deductive closure.

Tightness

Definition

A theory T is tight if any two deductively complete extensions of T in the same language are bi-interpretable iff they are identical.

(Without the same language restriction this is trivial. Consider e.g. ZF + "the new unary predicate is V" versus ZF + "the new unary predicate is $\emptyset$".)

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The following theories are both tight and semantically tight:

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- ZF (Enayat)
- Z_2, second-order arithmetic with full comprehension (Enayat)
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For example, ZFC + CH and ZFC + $\neg$CH are mutually interpretable: ZFC + CH can be interpreted as L, and ZFC + $\neg$CH can be interpreted through the boolean ultrapower approach to forcing.

But these interpretations lose information, and there is no way to produce a bi-interpretation.

The main question

For the nice foundational properties shared by ZF and KM, it's known that this requires the full strength of the theory.

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Question

Do we need the full strength of the theories to get tightness?

- Earlier work by Alfredo Roque Freire and Joel David Hamkins looked at certain fragments of ZF, showing they are not tight.
- Freire and I investigated fragments of KM, looking at GB and GB $+ \Sigma^1_k$-Comprehension.

The main theorem

Theorem (Freire–W.)

The following theories are not tight.

- GB;
- $\mathsf{GB} + \Sigma^1_k$-*Comprehension, for* $k \geq 1$.

GB is axiomatized by

- ZF for the sets;
- Extensionality for classes;
- Class Replacement—the image of a set under a class function is a set;
- Comprehension for first-order formulae—any class defined by a first-order formula must exist.

Σ^1_k-Comprehension says that classes defined by Σ^1_k-formulae must exist.

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After we started writing our paper, we learned that Ali Enayat had independently achieved this theorem in forthcoming work, using a different construction. (There's some technical details on what exactly his construction implies versus ours, with neither subsuming all of the other.)

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- But I think there's something of interest here outside of the small community of tight people!
- What does it mean for T to not be tight?
- It means we can find two different models of T, satisfying different theories, which are bi-interpretable.
- Indeed, we can do this in a uniform way.
- In this case we do this by showing that minimum models of class theories are bi-interpretable with carefully chosen Cohen extensions with the same sets.
- It seems to me that this kind of construction should be useful for other purposes, whether in set theory or second-order arithmetic.

A special case

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- We will look at models of class theories whose sets form V_κ for an inaccessible κ.
- We'll assume that $\mathrm{V}_\kappa \models \mathrm{V} = \mathrm{HOD}$, because we will need Skolem functions.
- I'll focus on the GB case, but I will gladly talk your ear off about the Σ^1_k-Comprehension case during a coffee break.

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T can't be defined over V_κ, but it can be defined over $(\mathrm{V}_\kappa, \mathcal{D})$:

- The Σ_k-truth predicate is definable via a Σ_k-formula.
- Being a partial truth predicate is recognizable by a first-order formula.
- You can express $\varphi[x] \in \mathrm{T}$ as "there exists a partial truth predicate which judges $\varphi[x]$ to be true". This is Σ^1_1.
- There's also a Π^1_1 definition: "every large enough partial truth predicate blah blah".
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- So in fact we can write down an axiom $\text{Class} = \mathcal{D}$ which says every class is definable.

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- Achtung! T needn't be an element of $\mathcal{X}$.
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- The strategy will be to interpret an extension by Cohen forcing $\mathrm{Add}(\kappa, 1)$.
- We'll find $\mathrm{C} \subseteq \mathrm{Add}(\kappa, 1)$ which is generic over $\mathcal{D}$ and definable over $\mathcal{D}$.
- Achtung! The definition necessarily will use class quantifiers!
- This will allow $\mathcal{D}$ to interpret $\mathcal{D}[\mathrm{C}]$.

Defining the Cohen generic C

From T you can define a κ-sequence of enough dense subsets of $\mathrm{Add}(\kappa, 1)$ to guarantee genericity over $\mathcal{D}$.

- Set D_α to consist of the intersection of the dense open sets definable from parameters in V_α.
- $\mathrm{Add}(\kappa, 1)$ is $<\kappa$-closed, so each D_α is dense open.
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Define C in κ many steps.

- At stage $\alpha + 1$, extend p_α to meet D_α.
- Use the HOD-order to choose $p_{\alpha+1}$. This is the only place we need the assumption $\mathrm{V}_\kappa \models \mathrm{V} = \mathrm{HOD}$!
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- Every $(\mathrm{V}_\kappa, \mathcal{X}) \models$ GB defines T the same, so they all define C the same.
- Because the forcing relations are definable, $\mathrm{T}(\mathrm{C})$ is definable from C. (This definition uses class quantifiers!)

Putting the interpretations together

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- To interpret $\mathcal{D}[C]$ in $\mathcal{D}$, use that $T[C]$ is definable in $\mathcal{D}$:

  Represent classes in $\mathcal{D}[C]$ by the HOD-least formula which defines them.
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Claim: $(V_\kappa, \mathcal{D})$ and $(V_\kappa, \mathcal{D}[C])$ satisfy different theories: they disagree on whether $\mathrm{Class} = \mathcal{D}$.

So we get bi-interpretable models of GB over V_κ which satisfy distinct theories.

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- The minimum model of Σ^1_k-CA over V_κ is obtained by building up $\mathrm{L}(\mathrm{V}_\kappa)$ below κ^+.
- Levels $\mathrm{L}_\alpha(\mathrm{V}_\kappa)$ are bi-interpretable with $\mathcal{L}_\alpha = \mathcal{P}(\mathrm{V}_\kappa) \cap \mathrm{L}_\alpha(\mathrm{V}_\kappa)$.
- And Σ_ℓ-formulae in $\mathrm{L}_\alpha(\mathrm{V}_\kappa)$ correspond to Σ^1_ℓ-formulae in $\mathcal{L}_\alpha$
- Let $\mathcal{D}_k = \mathcal{L}_\alpha$ for the minimum α to get a model of Σ^1_k-Comprehension.
- Use Jensen's Σ_ℓ-uniformization lemma to define a single subset T_k of V_κ which codes all of $\mathcal{D}_k$.

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This T_k controls $\mathcal{D}_k$ like how T controls $\mathcal{D}$.

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- The definitions aren't absolute to the same generality as for T and $\mathcal{D}$. But they are absolute between width extensions, and that's good enough for the bi-interpretation:
- $\mathcal{D}_k$ and $\mathcal{D}_k[\mathrm{C}]$ are bi-interpretable.

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- For this to work, we need Second-Order Replacement, a version of the Replacement axiom for functions defined using class quantifiers. This is enough to mimic the arguments that worked in the V_κ case.
- For example, over GB this guarantees that the Σ_k-truth predicate exists for every k, even nonstandard k.
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- This is a powerful axiom schema, but that just gives a stronger result: even a powerful extra axiom isn't enough to get tightness.
- For the second-order arithmeticians: In your context, we get that ACA and Π^1_k-CA—i.e. with full Induction—are non-tight, as opposed to just ACA_0 and Π^1_k-CA_0.

Thank you!

- Alfredo Roque Freire and Kameryn J. Williams, "Non-tightness in class theory and second-order arithmetic" (under review).
- Preprint: arXiv:2212.04445 [math.LO].