The universal algorithm, the universal finite sequence, and potentialism

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(they/them)

SHSU

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Overview

1. A look at a line of questions in my area,
2. Leading into where my work fits into this larger project,
3. With a little bit about my research as a whole.

A very accurate and nuanced history of the foundations of computation

Find an algorithm to solve the *Entscheidungsproblem**.

No.

* (Given a logical formula determine whether it is true in all structures.)

In a bit more detail

- The strategy to show an algorithm solves the *Entscheidungsproblem* is straightforward: exhibit the algorithm and check it does what you want.
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- Alonzo Church (1936), Alan Turing (1936), and others gave formalizations, which turn out to be equivalent.
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- Alonzo Church (1936), Alan Turing (1936), and others gave formalizations, which turn out to be equivalent.
- And since then there has been an explosion in equivalent characterizations, e.g. (an idealized version of) your favorite programming language.
- An advantage to giving a talk in 2023 is that computers are so ubiquitous I don't need to give you the formal definition of a Turing machine (TM).

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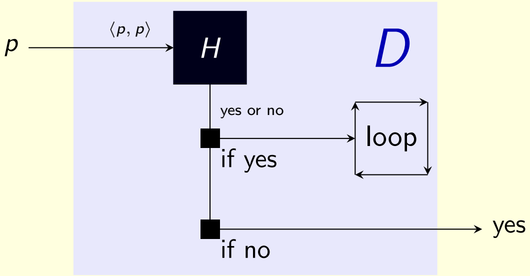
- Hard part! Turing showed that TMs are powerful enough to do computations involving other TMs. Indeed, he showed there is a universal machine which can simulate any TM.
- Other hard part! Turing's conceptual analysis to argue that his formalization correctly captures the intuitive notion of computability.
- Easy part! Do a diagonalization argument.

The easy part: the diagonalization argument

Toward a contradiction suppose H is a TM which decides whether or not p halts on input n. Let's build a new TM D.



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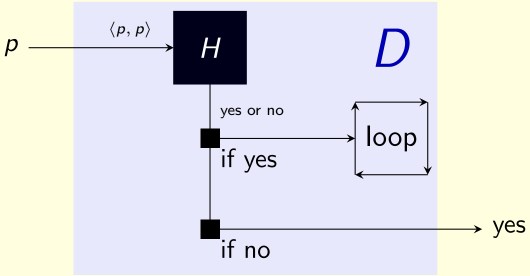

Now ask: what happens when D is input to D?

Then it halts iff it doesn't. ↯

From computability theory to proof theory

Let's talk about another kind of undecidability: proof theoretic, instead of computability theoretic.

And then we'll see how the two kinds of undecidability relate.

A very accurate and nuanced history of the incompleteness theorems

Find axioms that decide all questions of natural number arithmetic.

No.

The incompleteness theorems

Peano arithmetic (PA) axiomatizes natural number arithmetic: axioms of discretely ordered semirings + induction axioms.

Theorem (Gödel's first and second incompleteness theorems)

1. *No computably axiomatizable extension of* PA *is complete. There must be an arithmetic statement it neither proves nor disproves.*
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We need the restriction. True arithmetic TA—the set of all truths of $\mathbb{N}$—is complete.

(Moreover, the low basis theorem implies that there are complete extensions of PA which are arithmetically definable, specifically, Δ_2 in the arithmetical hierarchy.)

Arithmetization

- Gödel's beta lemma states that arbitrary finite sequences can be coded as a single number, and this is provable within PA.
- Thus any finite mathematical object can be coded in arithmetic.
  - What is a finite semiring? It's a tuple $\langle R, +, \times \rangle$ satisfying certain axioms. Represent R by a sequence of its elements and $+$ and $\times$ by sequences giving their multiplication tables. So you can write an arithmetic formula which expresses "n codes a finite semiring".

$+$	1	a	b	0
1	1	1	1	1
a	1	a	1	a
b	1	1	b	b
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$\times$	1	a	b	0
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  - More relevant to this talk, objects like Turing machines or logical formulae can be coded in arithmetic.
- Statements like "PA does not prove $0 = 1$" or "such and such Turing machine halts" can be cast as statements in arithmetic.

0 substituted for $[s]_0$, and φ with $[s]_0 + 1$ substituted for $[s]_0$. Now for the gory details. We define the relation $\mathrm{PA}(x)$, expressing that x is the Gödel-number of a Peano axiom by the formula

$$x = n_1 \vee \cdots \vee x = n_{15} \vee$$
$$\exists y, s \subseteq x \exists n \leq s \left(\begin{array}{l} \mathrm{Form}(y) \wedge \mathrm{len}(s) = n \wedge \\ \forall i < \mathrm{len}(n)\, \mathrm{Free}(y, [s]_i) \wedge \forall j \leq y(\mathrm{Free}(y, j) \to \exists k \leq s\, [s]_k = j) \wedge \\ \exists t \subseteq s \exists u, w \left(\begin{array}{l} \mathrm{len}(t) = \mathrm{len}(s) - 1 \wedge \forall i < \mathrm{len}(t)\, [t]_i = [s]_{i+1} \wedge \\ u = \mathrm{Sub}(y, [s]_0, \ulcorner 0 \urcorner) \wedge w = \mathrm{Sub}(y, [s]_0, \ulcorner [s]_0 + 1 \urcorner) \wedge \\ x = \ulcorner (\forall t\, (u \wedge (\forall [s]_0\, (y \to w) \to \forall [s]_0\, y))) \urcorner \end{array} \right) \end{array} \right).$$

(Taken with permission from Victoria Gitman's lecture notes for Mathematical Logic, Spring 2013.)

Incompleteness and Turing machines

The incompleteness theorems can be recast as saying that whether certain Turing machines halt is undecidable.

A TM p:

- Look at all length 1 proofs from the first 1 axiom of PA.
- Then look at all length 2 proofs from the first 2 axioms of PA.
- $\vdots$
- If at any point you see a proof that ends with $0 = 1$, halt and output affirmatively.

Whether p halts is independent of PA.

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- Adam Yedidia and Scott Aaronson do even better.
- They constructed a TM of size 7910 so that whether it halts is independent of ZFC, but ZFC + large cardinals does prove it halts.

(Specifically an ineffable cardinal will do.)

If you liked Gödel's incompleteness theorems, you'll love his completeness theorem

Theorem (Gödel's Completeness Theorem)

1. *A set of axioms T is consistent if and only if there is a structure satisfying T.*
2. *φ is a theorem of T if and only if φ is true in every structure satisfying T.*

(*This is for axioms in first-order logic.*)

- This lets us translate talk about proofs, consistency, etc. to talk about structures.
- The incompleteness theorems plus the completeness theorem together imply there must be non-isomorphic structures satisfying the axioms of arithmetic.

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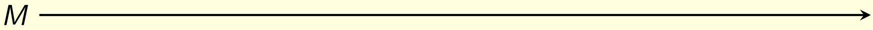
What could these even look like???

Nonstandard models of arithmetic

A model of (Peano) arithmetic is a discretely ordered semiring $(M, +, \times, <)$ whose definable subsets are inductive.



- $X \subseteq M$ is definable if you can express $x \in X$ just by quantifying over the elements of M and using the semiring operations and order of M.
- $X \subseteq M$ is inductive if $0 \in X$ and $a \in X \Rightarrow a + 1 \in X$ implies $X = M$.

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M has a least element 0^M ($=$ the additive identity for M) because the set $\{x \in M : x \geq 0^M\}$ satisfies the inductive hypotheses.

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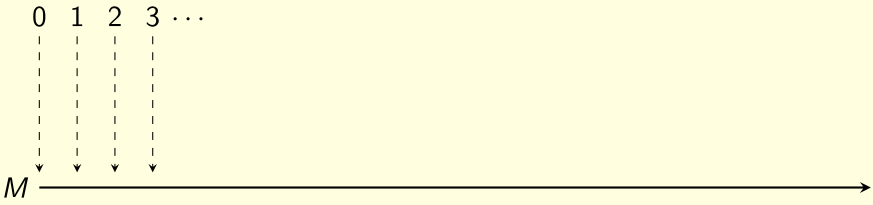

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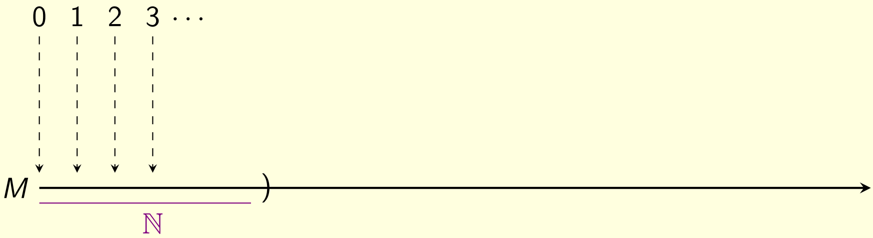

$\mathbb{N}$ embeds as an initial segment on any model of arithmetic.

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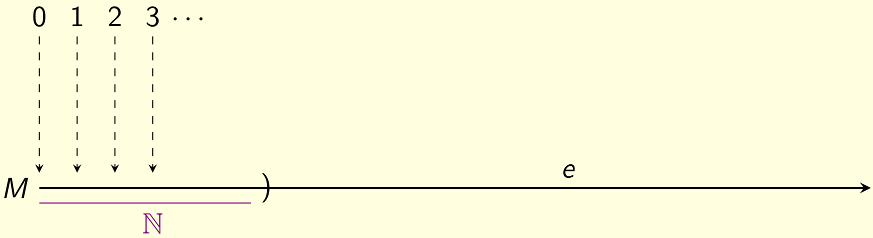

If $e \in M \setminus \mathbb{N}$ then $e > n$ for all $n \in \mathbb{N}$.

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All non-zero elements have a predecessor because

$$\{0\} \cup \{a \in M : a \text{ has a predecessor}\}$$

satisfies the induction hypotheses.

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A model of (Peano) arithmetic is a discretely ordered semiring $(M, +, \times, <)$ whose definable subsets are inductive.

$e + n < e + e = 2e$ for all $n \in \mathbb{N}$.

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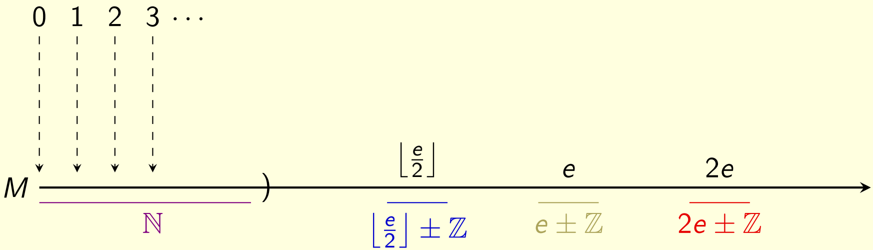

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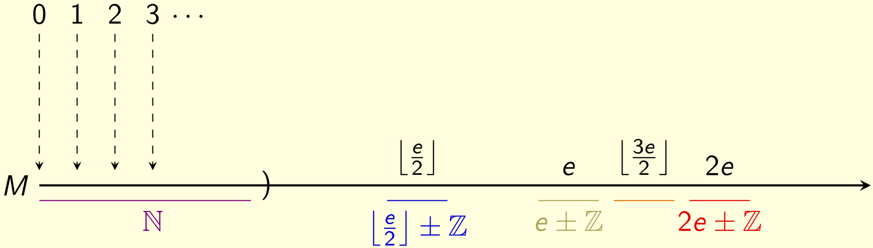

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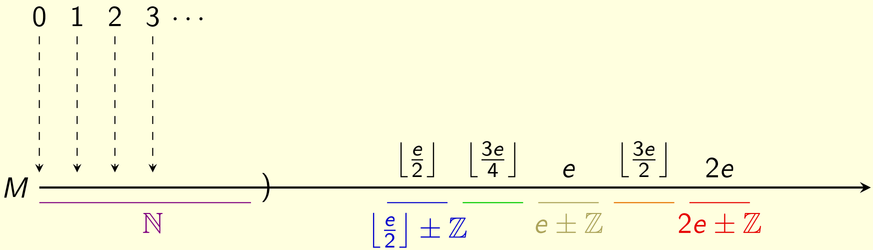

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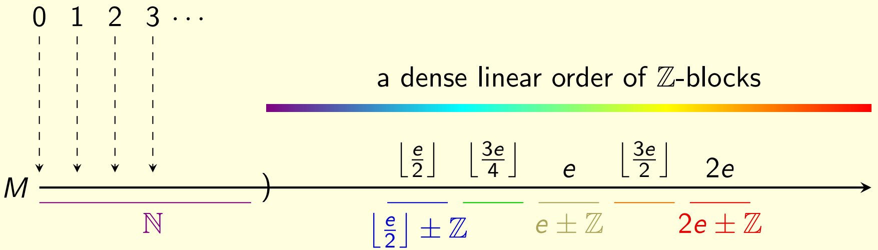

Facts about nonstandard models of arithmetic

- First constructed by Thoralf Skolem in the 1920s. (Skolem used an ultrapower construction.)
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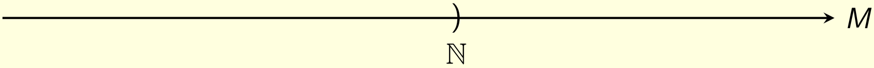
- Open Question (Harvey Friedman): $\mathbb{N}$ has the property that if a model of arithmetic is order-isomorphic to it then they are fully isomorphic. Does any other model of arithmetic have this property?
- (Stanley Tennenbaum) If M is nonstandard then neither the $+$ nor $\times$ of M is a computable function.

Turing machines in a nonstandard world



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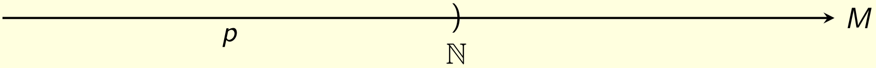

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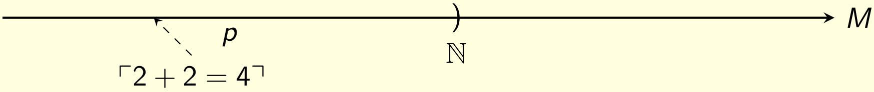

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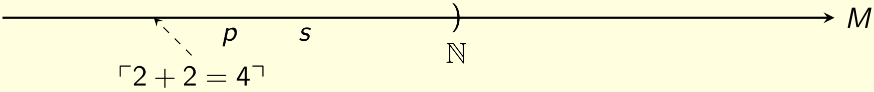

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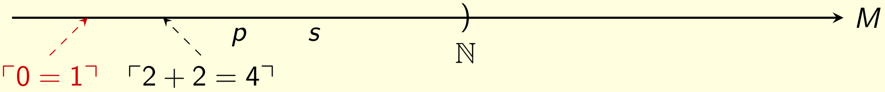

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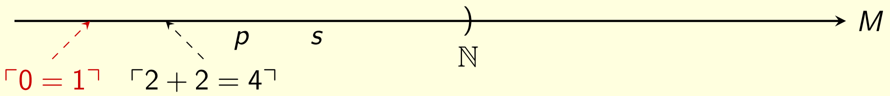

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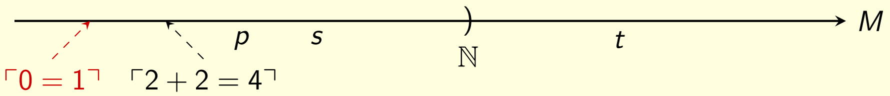

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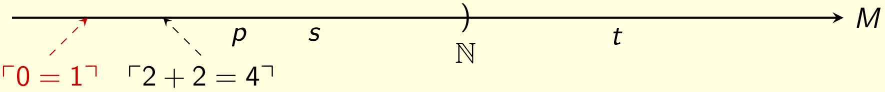

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- Then there is a computation log t witnessing that p outputs $\ulcorner 0=1 \urcorner$. But t must be nonstandard!
- The point: By moving to a larger world we made p output more numbers.

The absoluteness of computability

In summary:

- The statement "the TM p outputs n for some input" is upward absolute—if it's true it stays true if we end-extend to a larger model.

  Logicians call this a Σ_1 statement. (By the MRDP theorem, these are the statements equivalent to one whose only quantifiers are a block of $\exists$s.)

- But the statement "the TM p does not output n for some input" is not upward absolute. It is downward absolute though.

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  Peano arithmetic proves every true (i.e. in $\mathbb{N}$) statement of this form.
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  Both the first and second incompleteness theorems are about statements of this form.

Woodin's universal algorithm (2011)

We've seen that the behavior of a Turing machine can be undecidable.

- Proof theoretic: It may be independent of PA how p behaves.
- Model theoretic: Running p in different nonstandard models of arithmetic may produce different behavior.

I want to talk about a striking case of the undecidability of how Turing machines behave, due to W. Hugh Woodin, where p can output anything at all if run in the right universe!

Woodin's universal algorithm, first form

Theorem (Woodin)

There is a Turing machine p with the following properties.

1. *p provably enumerates a finite sequence.*
2. *Running p inside $\mathbb{N}$ never produces any output, i.e. it enumerates the empty sequence.*
3. *But, for any finite sequence s of natural numbers there is a nonstandard model of arithmetic M so that running p in M enumerates exactly s.*

Woodin's algorithm

(This construction for Woodin's theorem is due to Joel David Hamkins.)

The Turing machine p:

- p searches through the proofs of Peano arithmetic, looking at the theorems they prove.
- p is looking for a theorem of the form "p does **not** enumerate the sequence s", for s some nonempty finite sequence of numbers.

  (p can refer to itself by the Kleene recursion theorem.)
- If p ever sees this, then p outputs the sequence s.

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Claim: Run in $\mathbb{N}$, p outputs the empty sequence.

Otherwise p outputs some s. So Peano arithmetic proves this true Σ_1 statement. But by the definition of p, this also means that Peano arithmetic proves that p does not output s. This would mean that Peano arithmetic is inconsistent. But it's not.

Checking the extension property

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Claim: Peano arithmetic + "p outputs s" is consistent.

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Fix a finite sequence of natural numbers s. We want to find a nonstandard model of arithmetic M in which running p outputs s.

Claim: Peano arithmetic + "p outputs s" is consistent.

Otherwise "p does not output s" is a theorem of Peano arithmetic. But then running p in $\mathbb{N}$ would output a nonempty sequence. We just saw that is not the case.

Checking the extension property

Definition (The Turing machine p)

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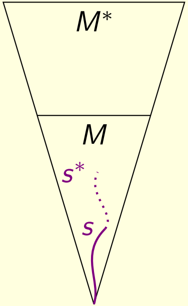
So by the completeness theorem there is a model of arithmetic in which p outputs s. □

Woodin's universal algorithm, general form

Theorem (Woodin)

There is a Turing machine p with the following properties.

1. *p provably enumerates a finite sequence.*
2. *Running p inside $\mathbb{N}$ never produces any output, i.e. it enumerates the empty sequence.*
3. *Suppose M a model of arithmetic in which p enumerates s and that s^* is a sequence in M which extends s. Then we can end-extend M to a larger model of arithmetic M^* in which p enumerates s^*.*



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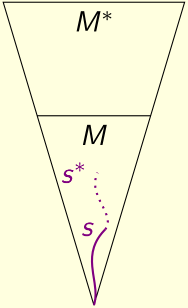
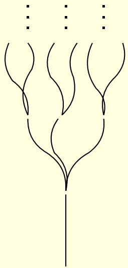


Proof idea: Do a similar argument, but internally to M. Need some more technical lemmata to check that the argument can be arithmetized.

A connection to philosophy: arithmetic potentialism



- Imagine climbing through the tree of nonstandard models of arithmetic, continually end-extending.
- This potentialist system gives a nonstandard twist on Aristotle's notion of the potential infinite.
- There is a natural interpretation in modal logic—extend ordinary logic by adding two new operators
  - $\Box\varphi$ means φ is necessarily true—true in all extensions.
  - $\Diamond\varphi$ means φ is possibly true—true in some extension.
- (Hamkins) Can use Woodin's universal algorithm to calculate which modal assertions are valid (true in any world under any substitution of variables).
  Namely, those in the modal theory S4.

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- Some of my research is in pure set theory, e.g. in aspects of Cohen's method of forcing and set-theoretic geology.
- The model theory of sets has been a recurring topic, from my dissertation work* to my most recent pre-print†.

* (Kameryn J. Williams, "Minimum models of second-order set theories", *The Journal of Symbolic Logic* (2019).)

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Is there a version of Woodin's universal algorithm for the world of infinitary mathematics?

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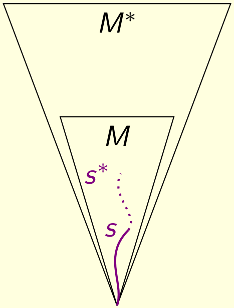
The universal finite sequence, basic version

Theorem (Hamkins–W.)

There is a Σ_1 definition for a finite sequence s so that:

1. ZF*, the basic axioms of set theory, proves s is a finite sequence.*
2. *If M is a well-founded model of* ZF *then its s is the empty sequence.*
3. *If M is a countable model of* ZF *with s as its sequence and s^* is any finite sequence in M extending s then there is a end-extension $M^* \models$* ZF *of M whose sequence is s^*.*



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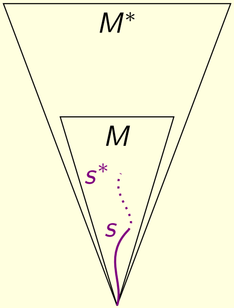

At core this is a similar sort of diagonal argument, but in a more difficult setting and needing more technology.

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Set-theoretic potentialism

Like how Woodin's universal algorithm enables an analysis of arithmetic potentialism, the universal finite sequence enables an analysis of set-theoretic potentialism.

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Corollary (Hamkins–W.)

The modal logic of end-extensional set-theoretic potentialism is exactly S4.

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- This is an interdisciplinary project: we're concerned with both mathematical results about these systems and how they inform philosophical debates within the philosophy of mathematics.

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Modal model theory, and student research

- Most of the extant work in potentialism has been on topics within logic.
- But this is a general framework.
- Suppose you have a class of mathematical structures, ordered by a substructure relation. (For example: groups, graphs, semirings.) Then there is an interpretation of modal logic in this context:

$$\Diamond\varphi \quad \text{is true at } M \text{ if } \varphi \text{ is true at some extension of } M$$
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- Last semester, I mentored an undergrad on a topic in this area, studying the modal model theory of special classes of graphs (triangle-free, etc.).

Applications of logic: nonstandard methods

- An application of model theory from Abraham Robinson (1960s).
- The idea: embed a mathematical structure M in a saturated elementary extension $^*M$. Facts about M carry upward to $^*M$, and the transfer principle lets you go in the other direction.
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- In an ongoing project with Tim Trujillo we've been looking using nonstandard methods in topological Ramsey theory.
- We have a nonstandard proof of the classical Nash-Williams partition theorem, and extensions of it to a more general setting.

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- **Future work:** More applications of nonstandard methods and ultrafilters to other areas of mathematics.
- **Future work:** More generally, I'm interesting in building connections between logic and other areas of mathematics.

Thank you!