A potential multiverse of classes

Kameryn J Williams
they/them

Sam Houston State University

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(Part of this talk is about joint work with Neil Barton.)

A puzzle from the early days of set theory

Hochverehrter Freund.

Wie ich Ihnen vor einer Woche schrieb, liegt mir viel daran, Ihr Urtheil in gewissen fundamentalen Puncten der Mengenlehre zu erfahren und bitte ich Sie, die Ihnen dadurch verursachte Mühe mir zu verzeihen.

Gehen wir von dem Begriff einer bestimmten Vielheit (eines Systems, eines Inbegriffs) von Dingen aus, so hat sich mir die Nothwendigkeit herausgestellt, zweierlei Vielheiten (ich meine immer *bestimmte* Vielheiten) zu unterscheiden.

Eine Vielheit kann nämlich so beschaffen sein, daß die Annahme eines „Zusammenseins" *aller* ihrer Elemente auf einen Widerspruch führt, so daß es unmöglich ist, die Vielheit als eine Einheit, als „ein fertiges Ding" aufzufassen. Solche Vielheiten nenne ich *absolut unendliche* oder *inconsistente Vielheiten.*

Wie man sich leicht überzeugt, ist z. B. der „Inbegriff alles Denkbaren" eine solche Vielheit; später werden sich noch andere Beispiele darbieten. [Anm. 1]

⋮

A. *Das System Ω aller Zahlen ist eine inconsistente, eine absolut unendliche Vielheit.*

Cantor in an 1899 letter to Dedekind

(Quoted from *Georg Cantor: Briefe*, pp. 408–409, eds H. Meschkowski & W. Nilson. 1991.)

A problem in the philosophy of set theory

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- The focus will be on understanding the mathematical structure of various philosophical answers, not on the philosophy per se.
- Specifically, I want to look at approaches which fit in the framework of set-theoretic potentialism.

But first, a warm-up!

To introduce the tools we'll use to study class-theoretic potentialism, I want to talk about an example with sets, due to Hamkins & Linnebo.

Zermelo's dynamic view of set

> Let us now put forth the general hypothesis that every categorically determined domain [V_κ, for κ inaccessible] can also be conceived of as a "set" in one way or another; that is, that it can occur as an element of a (suitably chosen) normal domain... Thus, to every categorically determined totality of "boundary numbers" [inaccessible cardinals] there follows a greater one, and the sequence of "all" boundary numbers is as unlimited as the number series itself... We must postulate the existence of an unlimited sequence of boundary numbers as a new axiom for the "meta-theory of sets".

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- Zermelo's view is sometimes reinterpreted in a universalist view for set theory.
- But it fits more naturally in a multiversalist view.
- Zermelo's multiverse is inflationist.
- There is a natural interpretation of modal logic in this context.

Modal Zermelo

Interpret Zermelo's view modally:

- Worlds are V_κ, for κ inaccessible.
- $V_\kappa \models \Diamond\varphi$ if there is $\lambda \geq \kappa$ so that $V_\lambda \models \varphi$.
- $V_\kappa \models \Box\varphi$ if $V_\lambda \models \varphi$ for *all* $\lambda \geq \kappa$.

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What is the modal logic of Zermelian potentialism?

In more detail:

- *Which propositional modal assertions are valid, i.e. true under any substitution of propositional variables for set theoretic formulae?*
- *Does this depend upon the world?*

A lower bound for Zermelian potentialism

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\begin{array}{ll}
(D) & \neg\Diamond p \Leftrightarrow \Box\neg p \\
(K) & \Box(p \Rightarrow q) \Rightarrow \Box p \Rightarrow \Box q \\
(T) & \Box p \Rightarrow p \\
(4) & \Box p \Rightarrow \Box\Box p \\
(.3) & (\Diamond p \wedge \Diamond q) \Rightarrow \Diamond([p \wedge \Diamond q] \vee [q \wedge \Diamond p])
\end{array}
$$

S4 is $(D + K + T + 4)$; S4.3 is S4 + (.3).

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S4 is $(D + K + T + 4)$; S4.3 is S4 + (.3).

Proof: S4 is valid for partially ordered frames, and (.3) is valid if the order is linear. □

An upper bound for Zermelian potentialism

Theorem (Hamkins & Linnebo)

S5, *i.e.* S4 + (5), *contains the modal validities for any world for Zermelian potentialism.*

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- The class of finite total relations is complete for S5.
- In particular, if ψ isn't in S5, there's some large enough total relation for which ψ is invalid.

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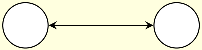
- The class of finite total relations is complete for S5.
- In particular, if ψ isn't in S5, there's some large enough total relation for which ψ is invalid.
- To prove this we need control statements which allow us to mimic the structure of total relations within Zermelian potentialism.

Control statements for Zermelian potentialism

Let's see an example:

$$p \Rightarrow \Box p \qquad \text{is not in S5.}$$

This can be witnessed by a two element frame:



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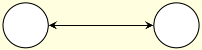

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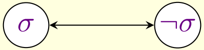

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Control statements for Zermelian potentialism

A switch is a statement σ so that $\Diamond\sigma$ and $\Diamond\neg\sigma$ are true at any world.

- A collection of switches are independent if any combination of their truth values can be freely toggled.
- (Hamkins & Löwe) If a potentialist system admits arbitrarily large finite families of independent switches then S5 is an upper bound for its modal validities.

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Let $\lambda + n$ denote the ordertype of the inaccessibles in the current world, where λ is either Ord or a limit ordinal and $n < \omega$.

This gives independent switches:

- σ_i: the ith bit of the binary expansion for n is 1.

More control statements for Zermelian potentialism

A button is a statement β which is possibly necessary; if β is true it is pushed, else it is unpushed.

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A long ratchet is a uniformly definable sequence $\langle \beta_\xi : \xi \in \mathrm{Ord} \rangle$ of buttons, so that pushing a button pushes all previous buttons on the sequence.

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If κ isn't a limit of inaccessibles, the modal validities at V_κ are exactly S4.3.

Exact calculations for Zermelian potentialism

A cardinal κ is 2-inaccessible if it is an inaccessible limit of inaccessibles.

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- If κ is 2-inaccessible + not a limit of 2-inaccessibles we get a long ratchet by asking how many 2-inaccessibles exist.
- But this doesn't work if κ is 3-inaccessible.
- If κ isn't 4-inaccessible, get a long ratchet by asking how many 3-inaccessibles exist.
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Do we eventually catch our tail?

Exact calculations for Zermelian potentialism

κ is Σ_3-reflecting if κ is inaccessible and V_κ is a Σ_3-elementary submodel of V.

(Using a definable Σ_3-truth predicate we can express this as a single assertion. Σ_3-reflecting cardinals exist if, for example, Ord is Mahlo.)

(The assertion "there are unboundedly many n-inaccessibles" is Π_3, and it follows that any Σ_3-reflecting cardinal is n-inaccessible, and more.)

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- (Hamkins & Linnebo) If κ is Σ_3-reflecting then the modal validities at V_κ are exactly S5.

Proof Sketch: We want to see $V_\kappa \models \Diamond\Box\varphi \Rightarrow \varphi$. So assume $V_\kappa \models \Diamond\Box\varphi$.
The statement "$\exists\alpha\ V_\alpha \models \Diamond\Box\varphi$" is a Σ_3-assertion in V, so you can apply Σ_3-reflection to get it inside V_κ, then reflection back upward yields $V_\kappa \models \varphi$.

Summarizing Zermelian potentialism

Theorem (Hamkins & Linnebo)

Under suitable large cardinal assumptions: The modal validities at any world for Zermelian potentialism are bounded below by S4.3 *and above by* S5*. Each bound is achieved exactly at certain worlds.*

Set-theoretic potentialism, in generality

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- (Hamkins & Linnebo) The modal validities at any world for countable transitive model potentialism are bounded below by S4.2 and above by S5. Each bound is achieved exactly at certain worlds.

  (Under the background theory ZFC + "every real is contained in a countable transitive model of ZFC".)

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Let's see a very different flavor of example.

Extending models of set theory

Let $M \subseteq N$ be models of set theory.

- N is an end-extension of M if $b \in M$ and $N \models a \in b$ implies $a \in M$. That is, N doesn't add new elements to objects in M.
- N is moreover a rank-extension of M if $b \in N \setminus M$ implies $\operatorname{rank} b \in N \setminus M$. That is, new elements are only added on top.

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What's new is M and N are not required to be well-founded. These notions make sense for ill-founded models.

- End-extensional potentialism has as worlds the countable models of set theory, ordered by end-extension.
- Rank-extensional potentialism has as worlds the countable models of set theory, ordered by rank-extension.

These are analogous to countable transitive model potentialism and Zermelian potentialism, but without a requirement that all worlds adhere to an external standard of well-foundedness.

Remark: If N end-extends well-founded M then any ill-foundedness in N must occur above the ordinals of M.

Set-theoretic potentialism allowing ill-founded worlds

Theorem (Hamkins & Woodin)

The modal validities at every world in rank-extensional potentialism are exactly S4.

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Remark: Trivially, S4 is a lower bound for any potentialist system—(T) expresses that the accessibility relation is reflexive and (4) expresses it is transitive.

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Both proofs follow a similar strategy: show that the potentialist system admits a universal finite sequence, a uniform definition for a finite sequence that can be freely extended by moving to the right larger world.

(These are set-theoretic analogues of Woodin's universal algorithm for models of arithmetic.)

A universal finite sequence gives control statements witnessing that S4 is an upper bound, using that the class of finite pre-trees is complete for S4.

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With rank-extensional and end-extensional potentialism:

- Statements like "the first entry on the universal sequence is ω_1" are possibly necessary.
- As we extend and commit to more and more of the universal sequence, we permanently close off the possibility of alternate realities where we instead put some other set next on the universal sequence.

Better understanding one's commitments

Taken together, these results show that the structure of modal truths for set theoretic potentialism depends upon whether there is a common standard of well-foundedness to which all worlds adhere.

Back to the main question

Having developed some tools, let's return to the question of what classes potentially could be.

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- We've already seen one possible answer, Zermelo's: classes are just sets in a larger world.
- But Zermelo isn't the only one to have an answer for this. Many mathematicians and philosophers have given answers to this question.

A popular—but insufficient—answer

Classes don't actually exist; talk of classes is just convenient shorthand for talk about (first-order) definable properties of sets.

- For example, "$\xi \in \mathrm{Ord}$" is shorthand for "ξ is transitive + linearly ordered by $\in$".

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The trouble is, there are uses of classes that cannot be captured just by looking at what is first-order definable.

Let's see an example.

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If the only classes are the definable classes, this is a triviality:

- If j is definable without parameters, then so is the critical point of j, the least ordinal moved by j. But any elementary embedding $V \to V$ must fix every definable object, so $j(\operatorname{crit} j) = \operatorname{crit} j$. $\lightning$
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If we think, as set theorists as a whole do, that there is substantive content to Kunen's theorem, it is in showing such j cannot even be an undefinable class.

What are classes then?

- Philosophers of mathematics and mathematicians have proposed different answers to what classes are, and how they differ from sets.
- Some of them admit a natural potentialist reading.
- (Barton & W.) Studying the mathematics of potentialism for sets can help us to better understand our commitments for what sets are. Perhaps the same can be done with potentialism for classes.

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- Philosophers of mathematics and mathematicians have proposed different answers to what classes are, and how they differ from sets.
- Some of them admit a natural potentialist reading.
- (Barton & W.) Studying the mathematics of potentialism for sets can help us to better understand our commitments for what sets are. Perhaps the same can be done with potentialism for classes.

Let's look at just one answer for what classes are.

Fujimoto's liberal predicativism

Developed by Fujimoto, following earlier work by Parsons.

Quote (Fujimoto 2019)

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- Classes are distinct from sets because they are part of language—predicates—unlike sets.
- But this goes beyond just definable classes. In particular, Fujimoto explicitly allows truth predicates as admissible predicates.
- He explicitly motivates his project with the need to allow talk of undefinable classes.

Abstracting Fujimoto

- The sets are a fixed totality.
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- (These abstract features are shared by other answers to what classes are, e.g. property theoretic approaches à la Linnebo or Fine.)

There is a Tarski-style hierarchy to truth predicates: truth about V, truth about truth, truth about truth about truth, and so on.

Iterated truth predicates are a device to put this hierarchy into a single class. Each iterated truth predicate has a length, which may be transfinite, and possibly even of length $> \mathrm{Ord}$.

Truth potentialist systems

Fix countable $M \models \mathrm{ZFC}$, to be the sets of the worlds. A truth potentialist system over M has worlds $(M, \mathcal{X})$ with classes $\mathcal{X}$ over M:

- Each world $(M, \mathcal{X})$ satisfies GB, namely predicative comprehension and class replacement.
- The definable classes of M form a world.
- If $(M, \mathcal{X})$ is a world and $A \in \mathcal{X}$, then there is a larger world containing the truth predicate relative to A as a parameter.

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- Modification: require there be larger worlds with iterated truth predicates relative to A of any length which exists. (This gives a modal version of Fujimoto's approach.)

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- Modification: require there be larger worlds with iterated truth predicates relative to A of any length which exists. (This gives a modal version of Fujimoto's approach.)
- More modification: require there be larger worlds with iterated truth predicates of length bounded by some Λ.

  ((E.g. $\Lambda = \mathrm{Ord}$ corresponds to Linnebo's approach.))

Basic facts about truth potentialist systems

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- The worlds are the classes definable from the length ξ iterated truth predicate (relative to no parameter), for different finite ξ.
- Ditto for requiring iterated truth predicates of bounded transfinite length. (Just need to allow longer lengths ξ, less than the bound.)
- And for requiring iterated truth predicates without bounds on their length. (You seem to need an extra technical condition here, about the worlds being correct about which classes are well-founded.)

The modal logic of truth potentialism

Theorem (Barton & W.)

Fix the sets M.

- *The smallest truth potentialist system for M validates* S4.3*, and ditto for the transfinite versions.*
- *If the lengths are unbounded or if the bound Λ is closed under addition $<\omega^2$, then the modal validities are exactly* S4.3*.*

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Proof sketch: For the first: observe this potentialist system is linearly ordered, similar to the Zermelian case.

For the second: we need a long ratchet.

β_ξ: "the length ξ iterated truth predicate (relative to no parameter) exists".

Then $\langle \beta_\xi : \xi < \Lambda \rangle$ is a long ratchet. The point is, to make Leibman's lemma (long ratchets give S4.3 as an upper bound) work, we only need that the length of the ratchet is closed under addition $<\omega^2$.

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- A global well-order can be added by a class forcing which doesn't add sets.

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Forcing to add a global well-order

The obvious thing works, and is equivalent to adding a Cohen-generic class of ordinals.

$\mathbb{G}$ consists of set-sized partial well-orders of the universe, ordered by end-extension.

- $\mathbb{G}$ is $<\kappa$-closed for every κ, so it doesn't add sets.
- An easy density argument yields that the generic well-orders all of V.

- A generic for $\mathrm{Add}(\mathrm{Ord}, 1)$ will, by density, code every set, and so gives a generic for $\mathbb{G}$ by ordering sets by where they are first coded.
- A generic for $\mathbb{G}$ gives a Cohen generic C by, say, putting $i \in C$ iff the ith set in the global well-order is an ordinal.

Truth potentialism with a global well-order

A class potentialist may want to say there's a (first-order) definable global well-order, but this has substantial cost. (It's equivalent to requiring that the sets satisfy $\exists x\ V = \mathrm{HOD}(\{x\})$.)

An alternative: instead of starting with a base world of the definable classes, start with a base world which contains a (possibly non-definable) global well-order. That is, the base world consists of all classes (first-order) definable from the global well-order.

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- The base world is $\mathrm{Def}(M, <)$ for some global well-order $<$.
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If the global well-order is sufficiently generic, these exist.

The modal logic of truth potentialism with a global well-order

- (Barton & W.) If the global well-order in the base world of this modification is sufficiently generic†, then the modal logic of this potentialist system is exactly S4.2.
  In particular, (.3) is invalid.
  This holds whatever lengths of iterated truth predicates you ask for.

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Proof sketch: any Cohen generic can be split into ω many mutual generics. By looking at the lengths of iterated truth predicates relative to these ω many pieces we get arbitrarily large families of independent buttons and switches, so S4.2 is an upper bound for the modal validities.

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Lemma (Killing Truth, W.)

Let M be a countable, transitive model of ZFC*. Then there is a Cohen-generic class C of ordinals so that C and the truth predicate for M cannot both be in the same* GB*-expansion for M. Indeed, from C and the truth predicate you can define a cofinal ω-sequence in the ordinals of M.*

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- The potentialist system consisting of all GB-expansions of M does not validate (.2).
- A modified truth potentialist system, with a new rule allowing extensions by adding a generic global well-order, will not validate (.2).

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Fix $B : \mathrm{Ord}^M \to 2$ such that the coordinates i where $B(i) = 1$ are cofinal and have ordertype ω.

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Claim: From the truth predicate you can define a sequence $\langle D_i : i \in \mathrm{Ord}^M \rangle$ of dense classes of $\mathrm{Add}(\mathrm{Ord}, 1)$ so that meeting all D_i guarantees genericity.

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Build C in Ord^M many stages: At stage $i+1$, extend with the minimal length to meet D_i, then add $B(i)$ as the next bit.

If you have both C and the truth predicate you can recover the coding points, and thereby define B. $\lightning$ $\square$

Class potentialism with global choice

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They are faced with multiple options on how to pay this cost:

- Put strong restrictions on the first-order theory of the sets.
- Accept that the modal structure of the multiverse is very different than the orderly structure of vanilla truth potentialism.
- Explicate a third option for global well-orders besides being definable or being generic.
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- Give up global choice, thereby giving up some interesting maths.
- Some other option I'm not clever enough to recognize.

A conjecture, and future work

Let T be a reasonable class theory, such as GB or KM and fix a countable model M of ZFC.

Conjecture

The potentialist system consisting of all T-expansions of M has exactly S4 *as its modal validities.*

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(This would address "top-down" approaches to class potentialism, based on the notion that reference to classes is indeterminate.)

Some evidence:

- The killing truth lemma implies S4.2 is too strong an upper bound for weak enough T.
- The analogous fact is true in second-order arithmetic, with full impredicative comprehension for 'classes'. (This is a corollary of the Hamkins & W. result about end-extensional potentialism.)
- For very strong T and ω-nonstandard M I can prove this. (But that is the least interesting instance of this conjecture...)

Thank you for listening!

Some references

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