Inner mantles: good, bad, and ugly

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(Partly joint work with Jonas Reitz)

0. Introduction
   A crash course in geology
1. The Good: Positive Results
2. The Bad: Negative Results

Geology 101

Question (Reitz): What if forcing, but backward?

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- Geology can be done in ZFC.
- All worlds in the generic multiverse are at most two steps away: M is a forcing extension of a ground of N.

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Open question: Are the grounds uniformly first-order definable over ZF?

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- (Gitman–Johnstone) If V is an extension of W by a poset of cardinality $\leq \delta$ and $W \models \mathrm{DC}_\delta$ then W is definable in V.
- (Usuba) If there is a proper class of Löwenheim–Skolem cardinals then the grounds are uniformly first-order definable.
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The bedrock axiom is true in, e.g., L while it is destroyed by set forcing.

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- The bedrock axiom $\mathrm{V} = \mathrm{M}$ asserts there are no nontrivial grounds.
- (Reitz) You can class force the bedrock axiom.

Do a set-support iteration of lottery sums to generically make the GCH fail/succeed at each regular cardinal.

Non-absoluteness and the mantle

- (Fuchs–Hamkins–Reitz) Every model of ZFC is the mantle of some class forcing extension.
- Thus, the mantle is not absolute. In particular, it is consistent that $\mathrm{M}^{\mathrm{M}} \neq \mathrm{M}$.

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- ($\mathrm{V} \subseteq \mathrm{M}^{\mathrm{V}[G]}$) By density, any set of ordinals in V is coded cofinally often into the GCH pattern of $\mathrm{V}[G]$, so any set of ordinals in V is in every ground of the extension.

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  Key point: grounds are correct about a tail of the GCH pattern.
- ($\mathrm{V} \supseteq \mathrm{M}^{\mathrm{V}[G]}$) Consider $x \in \mathrm{V}[G] \setminus \mathrm{V}$. The forcing can be factored into the product of a set-sized head and a sufficiently distributive tail so that the tail forcing could not add x. But then $\mathrm{V}[G^{\text{tail}}]$ is a ground which misses x.

Inner Mantles

The sequence of inner mantles M^i is defined inductively.

- $\mathrm{M}^0 = \mathrm{V}$;
- $\mathrm{M}^{i+1} = \mathrm{M}^{\mathrm{M}^i}$;
- $\mathrm{M}^\ell = \bigcap_{i<\ell} \mathrm{M}^i$ for limit ordinals ℓ.

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Observation: If M^i is a definable class, then M^{i+1} is a definable inner model of ZFC. So the only problem can be at limit stages.

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The Good: Consistently, the sequence of inner mantles can be as long as you like.

- **Theorem** (Reitz–W.):
  There is a class forcing notion, uniformly definable in η, which forces
  - The ground model is the η-th inner mantle; and
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- **Theorem** (W.):
  It can be that M^ℓ fails to be definable while M^i is definable for all $i < \ell$?
  It can be that M^ℓ is a definable model of $\neg$AC.
- (Compare to the classical theorems about iterated HOD by Harrington and McAloon.)

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The Ugly: To prove these theorems we will iterate forcings on orders which aren't well-orders.

0. Introduction
1. The Good: Positive Results (Joint with Reitz)
   Creationism for set theoretic geology
2. The Bad: Negative Results

Every model is the η-th inner mantle of another universe

Theorem (Reitz–W.)

There is a class forcing notion $\mathbb{M}(\eta)$, uniformly definable in a parameter $\eta \in \mathrm{Ord}$, so that forcing with $\mathbb{M}(\eta)$ produces a model $\mathrm{V}[G]$ satisfying

$$\mathrm{V} = (\mathrm{M}^{\eta})^{\mathrm{V}[G]}$$

where $\mathrm{M}^{i} \supsetneq \mathrm{M}^{i+1}$ for all $i < \eta$.

Overview of the proof

If η is finite, this is easy. Just repeatedly force with the Fuchs–Hamkins–Reitz partial order. Then you get $\mathrm{V}[\vec{G}] = \mathrm{V}[G_1] \cdots [G_\eta]$ satisfying

- $(\mathrm{M}^1)^{\mathrm{V}[\vec{G}]} = \mathrm{V}[G_1] \cdots [G_{\eta-1}]$;
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- $\vdots$
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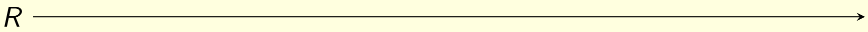
The problem: the order of the inner mantles reverses the order of the iteration. For infinite η, we want to force with an $\eta^\star$-iteration of class products, not an η-iteration.

Set theorists do not have a general theory of iterations on ill-founded orders. But we can handle this specific case.

Defining the forcing $\mathbb{M}(\eta)$



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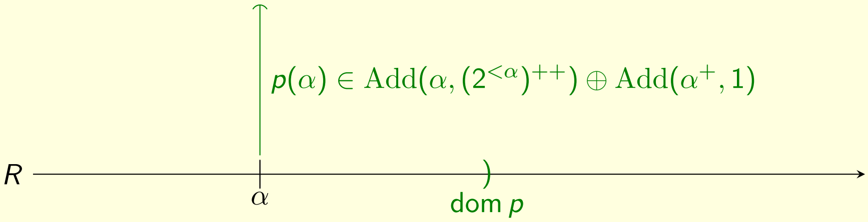

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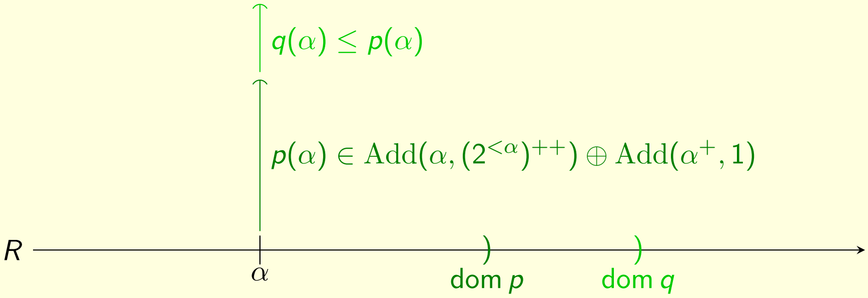

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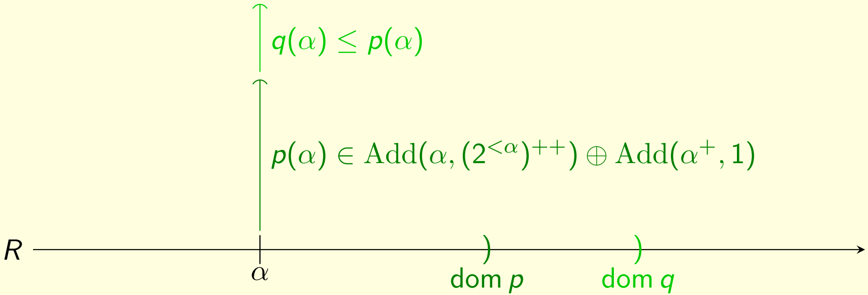

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- Split R into congruence classes R_i for $i < \eta$.
- Then $\langle R_{>i} : i \in \eta \rangle$ is a $\subsetneq$-descending sequence of ordertype η.
- For $\alpha \in R$ let $i(\alpha)$ be the unique i with $\alpha \in R_i$.

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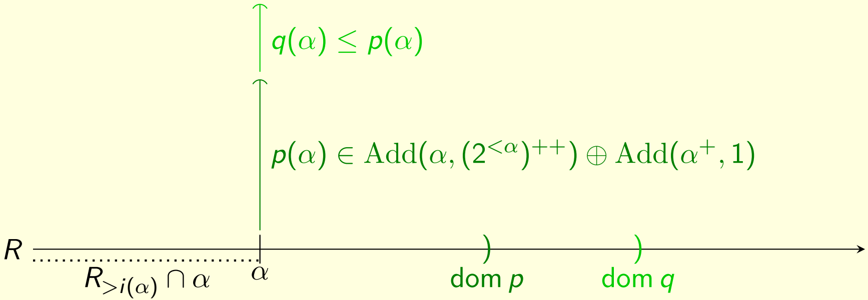

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- For $\alpha \in R$ let $i(\alpha)$ be the unique i with $\alpha \in R_i$.
- $p(\alpha)$ is a $\mathbb{M}(\eta) \upharpoonright (R_{>i(\alpha)} \cap \alpha)$-name for an appropriate condition.
- $p \upharpoonright (R_{>i(\alpha)} \cap \alpha)$ forces over $\mathbb{M}(\eta) \upharpoonright (R_{>i(\alpha)} \cap \alpha)$ that $q(\alpha) \leq p(\alpha)$.

Defining the forcing $\mathbb{M}(\eta)$

Fix a suitable coding region R. Split R into η many congruence classes R_i. For $\alpha \in R$ let $i(\alpha)$ be the unique $i < \eta$ so that $\alpha \in R_i$. Let $R_{>i}$ have the obvious meaning.

$\mathbb{M}(\eta)$ is the class forcing

- whose conditions are set-sized functions p with domain an initial segment of R
- so that for all $\alpha \in \operatorname{dom} p$ we have $p(\alpha)$ is an $\mathbb{M}(\eta) \restriction (R_{>i(\alpha)} \cap \alpha)$-name for a condition in $\mathrm{Add}(\alpha, (2^{<\alpha})^{++}) \oplus \mathrm{Add}(\alpha^+, 1)$.
- For $p, q \in \mathbb{M}(\eta)$, say that $q \leq p$ if
  - $\operatorname{dom} q \supseteq \operatorname{dom} p$ and
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For later purposes we will need $\mathbb{M}(\eta)$ to be η^+-distributive. This is easily arranged by having R only contain cardinals $\geq \eta^+$.

Questions about $\mathbb{M}(\eta)$

- $\mathbb{M}(\eta)$ was defined as a weird iteration of ordertype Ord. In what sense can we think of it as an iteration of ordertype $\eta^\star$?
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- $\mathbb{M}(\eta)$ is a progressively distributive iteration: for $\alpha \in R$, $\mathbb{M}(\eta)$ factors as $\mathbb{M}(\eta) \cong \mathbb{M}(\eta)_{\mathsf{head}} * \mathbb{M}^{\mathsf{tail}}$ where $\mathbb{M}(\eta)_{\mathsf{head}} \Vdash \mathbb{M}^{\mathsf{tail}}$ is α-distributive.
- In particular, $\mathbb{M}(\eta)$ preserves ZFC.
- $\mathbb{M}(\eta)$ preserves R and each R_i.
- The same holds for $\mathbb{M}(\eta) \restriction R_{\geq i}$.

$\mathbb{M}(\eta)$ as an $\eta^\star$-iteration

For notational convenience: set $\mathbb{P} = \mathbb{M}(\eta)$ and $\mathbb{P}_i = \mathbb{M}(\eta) \upharpoonright R_{\geq i}$.

Observation

$$\mathbb{P} = \mathbb{P}_0 \supseteq \mathbb{P}_1 \supseteq \cdots \supseteq \mathbb{P}_i \supseteq \cdots \supseteq \mathbb{P}_\eta \qquad i \leq \eta$$

is a continuous descending chain of class forcing notions, and for $i < j$ we have $\mathbb{P}_j$ is a complete suborder of $\mathbb{P}_i$.
In particular, $\mathbb{P}$ factors as $\mathbb{P}_i * \dot{\mathbb{Q}}^{\text{tail}}$ for each $i < \eta$.

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Let $G \subseteq \mathbb{P}$ be generic over V, and let G_i be the restriction of G to $\mathbb{P}_i$. In particular G_η is the trivial filter over the trivial forcing $\mathbb{P}_\eta$.

Claim

For $i \leq \eta$, $(\mathrm{M}^i)^{\mathrm{V}[G]} = \mathrm{V}[G_i]$.

Prove this by induction.

The successor step

$\mathbb{P}_i$ factors as $\mathbb{P}_{i+1} * \dot{\mathbb{Q}}_i$ where

$$\mathbb{Q}_i = \prod_{\alpha \in R_i} \mathrm{Add}(\alpha, (2^{<\alpha})^{++}) \oplus \mathrm{Add}(\alpha^+, 1).$$

Now do the Fuchs–Hamkins–Reitz argument.

The limit step

Lemma (Jech)

Let i be a limit ordinal and

$$\mathbb{B}_0 \supseteq \mathbb{B}_1 \supseteq \cdots \supseteq \mathbb{B}_j \supseteq \cdots \supseteq \mathbb{B}_i$$

be a continuous descending sequence of complete sub-boolean algebras, where $\mathbb{B}_0$ is i^+-distributive. If $G_0 \subseteq \mathbb{B}_0$ is generic over V and $X \in V[G_j]$ for all $j < i$, then $X \in V[G_i]$.

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But if $\mathbb{P}$ is a progressively distributive iteration, factoring as $\mathbb{Q}_\alpha * \mathbb{Q}^{\text{tail}}$ for arbitrarily large α so that the $\mathbb{P}_j \cap \mathbb{Q}_\alpha$ form a chain like in Jech's lemma, then we get the conclusion of Jech's lemma.

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This is where we use the assumption that $\mathbb{P} = \mathbb{M}(\eta)$ is η^+-distributive!

Aside: what is iterated HOD in $V^{\mathbb{M}(\eta)}$?

- **Theorem** (Reitz–W.) In the forcing extension by $\mathbb{M}(\eta)$, the sequences of inner mantles and iterated HOD exactly line up: $\mathrm{M}^i = \mathrm{HOD}^i$ for all $i \leq \eta$.

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Yes we can. Fuchs, Hamkins, and Reitz give forcings to separate the mantle and HOD, and we can build on them to separate iterated HOD and inner mantles.

- **Theorem** (Reitz–W.) Fix an ordinal η. There is a forcing which forces the ground model to be the η-th inner mantle while forcing the extension to be its own HOD.
- And there is another forcing which forces the ground model to be the η-th iterated HOD while forcing the extension to be its own mantle.

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Combining these forcings with $\mathbb{M}(\eta)$ we can make the sequence of iterated HODs and inner mantles have any two lengths we wish, with one as an initial segment of the other.

Aside: open questions on separating the two sequences

Can we more finely control how to separate the two sequences?

Question

Let η be an ordinal. Can we force the sequence the ground model to be M^η and HOD^η of the extension, but $\mathrm{M}^i \neq \mathrm{HOD}^i$ for all $0 < i < \eta$? Can we moreover get $\mathrm{M}^i \neq \mathrm{HOD}^j$ for all $0 < i, j < \eta$?

Question

Let η be an ordinal. Can we force the sequence of inner mantles to have length η so that $\mathrm{M}^i = \mathrm{HOD}^{2i}$ for all $i \leq \eta$? What about vice versa? What if we replace 2 with a different ordinal?

Set theoretic arbology

To cause problems at the limit stage we will need to precisely control what sets get into which inner mantles. For this we will use what I call tree iterations.

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- In $\mathrm{L}[x][\bar{c}]$: we can recover x in any ground by looking at the Cohen pattern in $[\aleph_{\omega\cdot\xi}, \aleph_{\omega\cdot\xi+\omega})$ for some large enough ξ. So $x \in \mathrm{M}$.

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To get x into M^2 we'd want to in turn code each $c_{\omega\cdot\xi+n}$ into the Cohen pattern cofinally often, and so on to get even deeper.

So there's a tree-like structure to the order of the coding.

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Self-encoding forcing to code x is an ω-iteration of Cohen forcings.

- First code x, get a block of Cohens $\vec{c}_1$.
- Given a block $\vec{c}_n$ of Cohens, code them, getting a block of Cohens $\vec{c}_{n+1}$.

In any extension where you didn't add any new Cohens to the coding region, the generic $\vec{c} = \langle \vec{c}_n : n \in \omega \rangle$ is definable, and this definition is uniform using only the coding region as a parameter.

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- Our inner mantles will be generated by proper classes of Cohens, so to ensure we can correctly line up each inner mantle with its corresponding class of Cohens we will use self-encoding forcing.
- To avoid coding more than we want, at each coding point we will use $\mathrm{Add}(\alpha, 1)$ as defined in L. This kills closure properties, but by general facts about generalized Cohen iterations we still have distributivity.

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Rather than do a linear iteration, we want to do an iteration $\mathbb{P}$ along a tree T.

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For my context:

- All trees are well-founded.
- All supports are set-support.

Technical lemmata

Non-linear iterations have been studied before, e.g. by Jech and Groszek (1991). Specializing some of their work about distributivity and chain conditions to my context, we get:

- **Safety Lemma:** Consider a tree iteration along T, where for each cardinal κ there is at most one stage $s \in T$ which adds a Cohen subset to κ. Then, the only Cohen subsets of κ are those added by stage s.
- This is why I use set-support everywhere!

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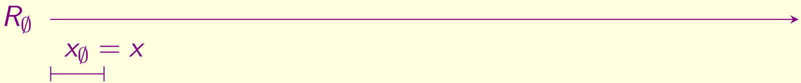

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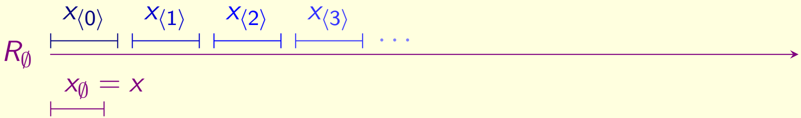

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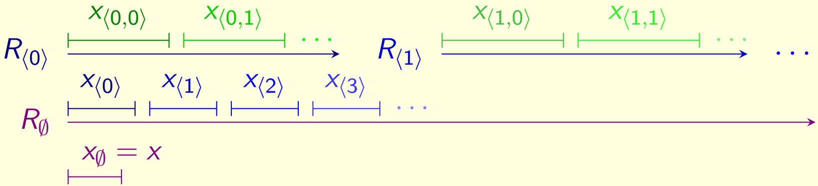

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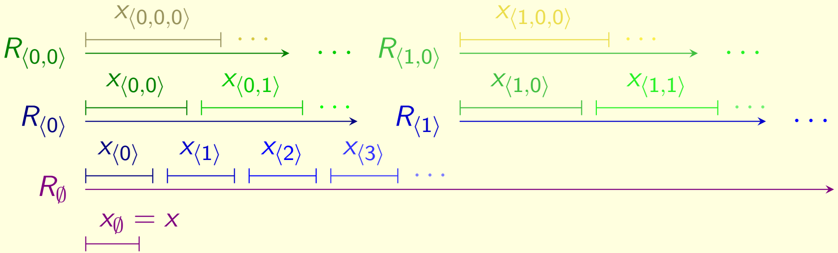

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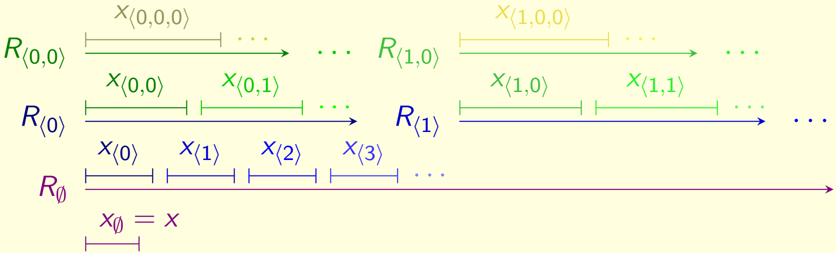

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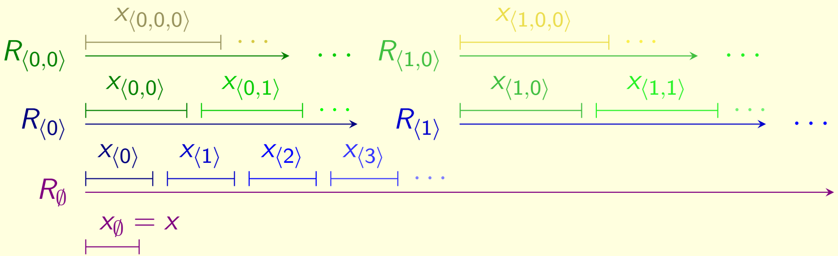

Call this tree-like coding by $\mathbb{T}_k(x)$ for short.

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- In particular, $A^k \in \mathrm{M}^k \setminus \mathrm{M}^{k+1}$

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- So the class-forcing extension for the theorem cannot have M^ω as a definable class, as then it could define the truth predicate for L.

More detail on coding L-truth into M^ω

Idea to prove the theorem (Harrington): Assign to each formula φ with parameters from L a cardinal $\kappa(\varphi)$. Then code so that M^ω has a Cohen subset of $\kappa(\varphi)$ iff $\mathrm{L} \models \varphi$. We need to define the coding forcing using only a bounded level of truth in L to ensure that the forcing is definable.

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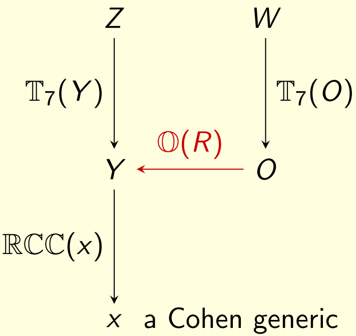
For this we will need more coding tools.

- You can code a proper class X, say by definably breaking X into set-sized chunks and coding the chunks on definable subregions.
- If you tweak self-encoding forcing to have Ord many stages instead of ω many, then not only will the full generic be definable but also it will remain so in every ground, whence the coded set will get into all inner mantles.
  Call this robust Cohen coding $\mathbb{RCC}(x)$.
- You can overwrite a coding block R by adding a Cohen generic to every $\alpha \in R$. Let $\mathbb{O}(R)$ be the overwrite forcing for R.

  It may be that the original codes in R are still definable—e.g. if they were coded elsewhere—but you can use overwrite forcing to erase coded information.

A toy example of more complicated coding



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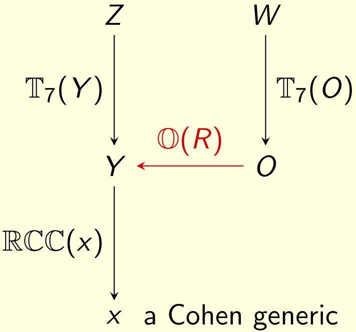

The code W ensures O survives to M^7, overwriting the region R where the code Y lives. Nevertheless, before we dig past M^7 we can recover Y using the code Z. Namely, Z ensures that Y is in M^7, which in turn ensures that $x \in \mathrm{M}^8$. But in M^7 we no longer have a code for Y, and the coding region was overwritten. So in M^8 we have that x is no longer Cohen coded, and thus $x \notin \mathrm{M}^9$.

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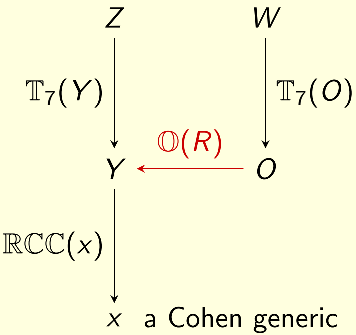

Why not just use $\mathbb{T}_8(x)$?

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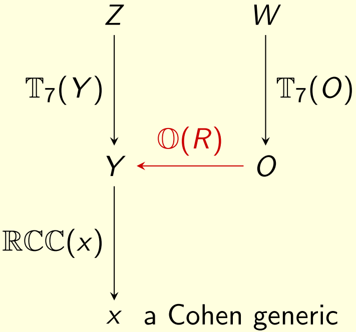
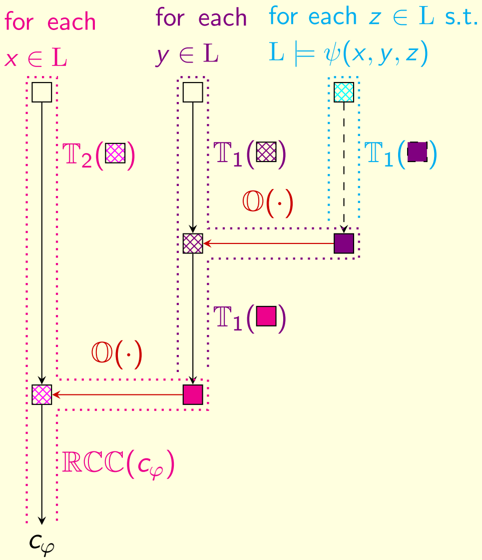


Why not just use $\mathbb{T}_8(x)$?

The point: O is what kept x out of M^9. If we had in turn overwritten the code W for O then we would've ensured $x \in \mathrm{M}^\omega$.

The code W ensures O survives to M^7, overwriting the region R where the code Y lives. Nevertheless, before we dig past M^7 we can recover Y using the code Z. Namely, Z ensures that Y is in M^7, which in turn ensures that $x \in \mathrm{M}^8$. But in M^7 we no longer have a code for Y, and the coding region was overwritten. So in M^8 we have that x is no longer Cohen coded, and thus $x \notin \mathrm{M}^9$.

Triangle coding—do for every φ with parameters from L



- $\varphi = \exists x \neg \exists y \exists z \ \psi(x, y, z)$ is Σ_3.
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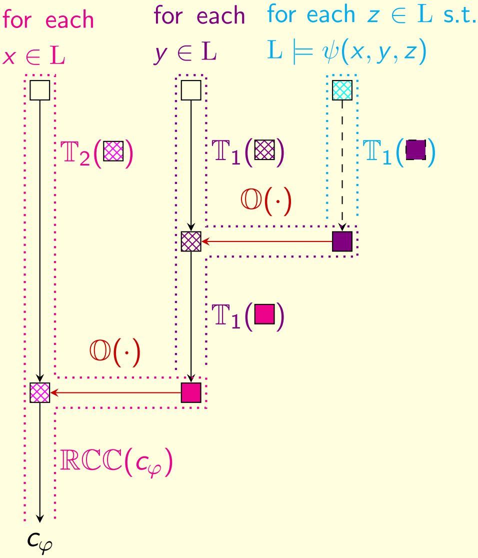

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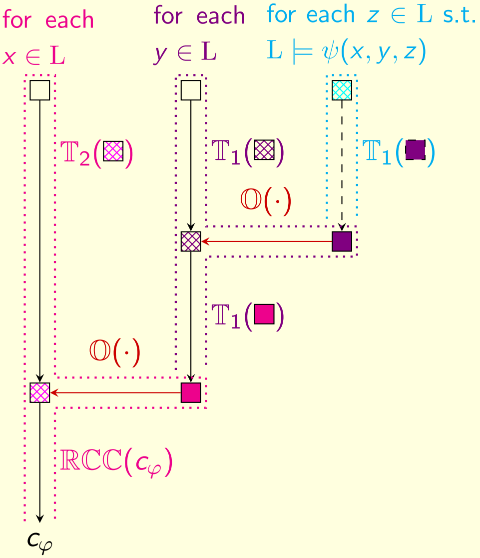

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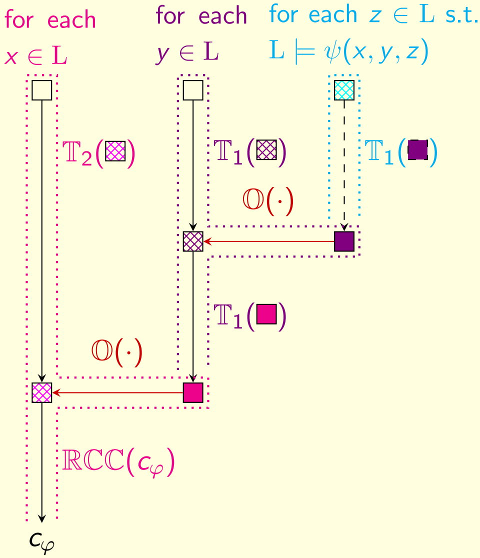

- $\varphi = \exists x \neg \exists y \exists z\ \psi(x, y, z)$ is Σ_3.
- $c_\varphi \subseteq \kappa(\varphi)$ is Cohen generic.
- Always ▧ survives to M^2 and ▧ survives to M^1.
- Fix $x \in \mathrm{L}$. We want to use ▧ to get c_φ is in M^ω, but in M^2 ▧ is overwritten if ■ survives.
- ■ survives into M^2 iff for some y ▧ is not overwritten in M^1.

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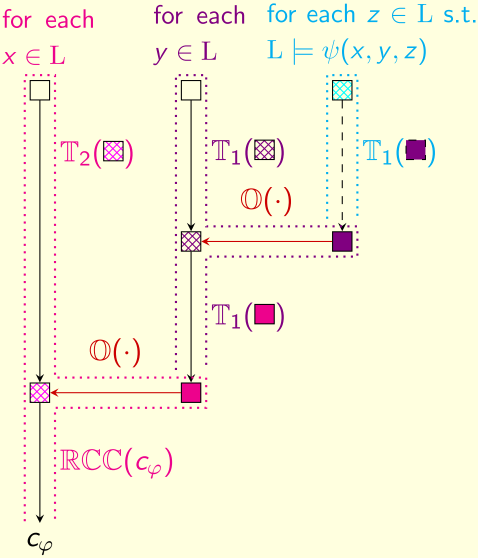

- $\varphi = \exists x \neg \exists y \exists z\ \psi(x, y, z)$ is Σ_3.
- $c_\varphi \subseteq \kappa(\varphi)$ is Cohen generic.
- Always ⊠ survives to M^2 and ⊠ survives to M^1.
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- ■ survives into M^2 iff for some y ⊠ is not overwritten in M^1.
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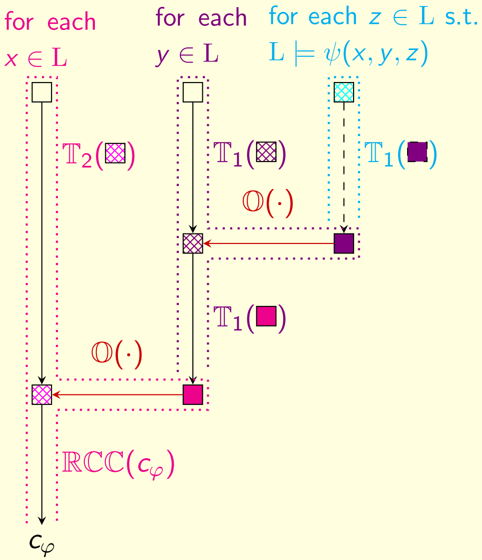

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- Always ▩ survives to M^2 and ▩ survives to M^1.
- Fix $x \in \mathrm{L}$. We want to use ▩ to get c_φ is in M^ω, but in M^2 ▩ is overwritten if ■ survives.
- ■ survives into M^2 iff for some y ▩ is not overwritten in M^1.
- ▩ is overwritten in M^1 iff ■ gets into M^1 iff for some z ▩ exists.
- Altogether: c_φ gets into M^ω iff $\exists x \in \mathrm{L}$ so that $\neg \exists y \in \mathrm{L}$ so that $\exists z \in \mathrm{L}$ so that $\mathrm{L} \models \psi(x, y, z)$.

A remaining question: is ω special?

- For the negative results I showed that M^ω can be badly behaved.
- Can we get the same for other limit stages?

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- For the negative results I showed that M^ω can be badly behaved.
- Can we get the same for other limit stages?
- We need a way to ensure a set survives into transfinite stage inner mantles.

Stretching branches

- Start by considering the Reitz–W. forcing to make V the η-th inner mantle of the extension.
- You can modify this forcing to get a forcing which ensures x gets into the η-th inner mantle, and that the sequence doesn't stabilize before η.
- Also, you can modify it to use Cohen coding instead of continuum coding.
- Call this $\mathbb{S}_\eta(x)$.

- (Instead of coding all of V just code x. You can't use density to pick where to code, since that codes too much, so you need to do the bookkeeping by hand.)

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- Call this $\mathbb{S}_\eta(x)$.
- By placing an $\mathbb{S}_\eta$ before a subtree in a tree iteration, you can get that x survives into M^η, not just finite stages.

- (Instead of coding all of V just code x. You can't use density to pick where to code, since that codes too much, so you need to do the bookkeeping by hand.)

Limit stages may not satisfy AC

Theorem (W.): Let η be a limit ordinal. There is a class forcing extension of L in which the η-th inner mantle M^η is a definable inner model of ZF in which $\mathcal{P}(\operatorname{cof}\eta)$ cannot be well-ordered.

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- Let $\lambda = \operatorname{cof}\eta$ and fix a sequence $\langle \eta_i : i < \lambda \rangle$ cofinal in η.

- Add λ^+ many Cohen subsets of λ.
- Call the block of Cohens by A, and let A^i denote the block of tails of the Cohens from i on.
- Similar to the ω case but using $\mathbb{S}_{\eta_i}$s, do a tree iteration to ensure A^i gets into M^{η_i} but not into M^{η_i+1}.

The rest of the argument goes through like the ω case.

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Theorem (W.): Let η be a limit ordinal with countable cofinality. There is a class-forcing extension of L whose M^η can define the truth predicate for L.

- Fix a sequence $\langle \eta_n : n < \omega \rangle$ cofinal in η.
- Using $\mathbb{S}_{\eta_n}$ and triangle coding, you can ensure $c_\varphi \subseteq \kappa(\varphi)$ always survives into M^{η_n} but only survives past $\mathrm{M}^{\eta_n + n}$ if $\mathrm{L} \models \varphi$. (Where φ is Σ_n.)
- So M^η can define truth for L by querying whether $\kappa(\varphi)$ has a Cohen subset.

Thank you!

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