Potentialism about sets, potentialism about classes

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Zermelo's dynamic view of set

> Let us now put forth the general hypothesis that every categorically determined domain [V_κ, for κ inaccessible] can also be conceived of as a "set" in one way or another; that is, that it can occur as an element of a (suitably chosen) normal domain... Thus, to every categorically determined totality of "boundary numbers" [inaccessible cardinals] there follows a greater one, and the sequence of "all" boundary numbers is as unlimited as the number series itself... We must postulate the existence of an unlimited sequence of boundary numbers as a new axiom for the "meta-theory of sets".

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Modal Zermelo

Interpret Zermelo's view modally:

- Worlds are V_κ, for κ inaccessible.
- $V_\kappa \models \Diamond\varphi$ if there is $\lambda \geq \kappa$ so that $V_\lambda \models \varphi$.
- $V_\kappa \models \Box\varphi$ if $V_\lambda \models \varphi$ for *all* $\lambda \geq \kappa$.

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Question

What is the modal logic of Zermelian potentialism?

In more detail:

- *Which propositional modal assertions are valid, i.e. true under any substitution of propositional variables for set theoretic formulae?*
- *Does this depend upon the world?*

A lower bound for Zermelian potentialism

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\begin{array}{rl}
(D) & \neg\Diamond p \Leftrightarrow \Box\neg p \\
(K) & \Box(p \Rightarrow q) \Rightarrow \Box p \Rightarrow \Box q \\
(S) & \Box p \Rightarrow p \\
(4) & \Box p \Rightarrow \Box\Box p \\
(.3) & (\Diamond p \wedge \Diamond q) \Rightarrow \Diamond([p \wedge \Diamond q] \vee [q \wedge \Diamond p])
\end{array}
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S4 is $(D + K + S + 4)$; S4.3 is S4 $+ (.3)$.

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S4 is $(D + K + S + 4)$; S4.3 is S4 $+ (.3)$.

Proof: S4.3 is valid for linearly ordered frames, and the V_κs are linearly ordered. □

An upper bound for Zermelian potentialism

Theorem (Hamkins & Linnebo)

S5, *i.e.* S4 + (5), *contains the modal validities for any world for Zermelian potentialism.*

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- The class of finite total relations is complete for S5.
- In particular, if ψ isn't in S5, there's some large enough total relation for which ψ is invalid.

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Upper bounds are harder than lower bounds.

- The class of finite total relations is complete for S5.
- In particular, if ψ isn't in S5, there's some large enough total relation for which ψ is invalid.
- To prove this we need control statements which allow us to mimic the structure of total relations within Zermelian potentialism.

Control statements for Zermelian potentialism

A switch is a statement σ so that $\Diamond\sigma$ and $\Diamond\neg\sigma$ are true at any world.

- A collection of switches are independent if any combination of their truth values can be freely toggled.
- (Hamkins & Löwe) If a potentialist system admits arbitrarily large finite families of independent switches then S5 is an upper bound for its modal validities.

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Let $\lambda + n$ denote the ordertype of the inaccessibles in the current world, where λ is either Ord or a limit ordinal and $n < \omega$.

This gives independent switches:

- σ_i says the ith bit of the binary expansion for n is 1.

More control statements for Zermelian potentialism

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A long ratchet is a uniformly definable sequence $\langle \beta_\xi : \xi \in \mathrm{Ord} \rangle$ of buttons, so that pushing a button pushes all previous buttons on the sequence.

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If κ isn't a limit of inaccessibles, the modal validities at V_κ are exactly S4.3.

Exact calculations for Zermelian potentialism

A cardinal κ is 2-inaccessible if it is an inaccessible limit of inaccessibles.

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Can we always pull this kind of trick?

Exact calculations for Zermelian potentialism

κ is Σ_3-reflecting if κ is inaccessible and V_κ is a Σ_3-elementary submodel of V.

(Using a definable Σ_3-truth predicate we can express this as a single assertion. Σ_3-reflecting cardinals exist if, for example, Ord is Mahlo.)

(The assertion "there are unboundedly many n-inaccessibles" is Π_3, and it follows that any Σ_3-reflecting cardinal is n-inaccessible, and more.)

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- (Hamkins & Linnebo) If κ is Σ_3-reflecting then the modal validities at V_κ are exactly S5.

Proof Sketch: We want to see $V_\kappa \models \Diamond\Box\varphi \Rightarrow \varphi$. So assume $V_\kappa \models \Diamond\Box\varphi$.
The statement "$\exists\kappa\ V_\kappa \models \Diamond\Box\varphi$" is a Σ_3-assertion in V, so you can apply Σ_3-reflection to get it inside V_κ, then reflection back upward yields $V_\kappa \models \varphi$.

Summarizing Zermelian potentialism

Theorem (Hamkins & Linnebo)

Under suitable large cardinal assumptions: The modal validities at any world for Zermelian potentialism are bounded below by S4.3 *and above by* S5*. Each bound is achieved exactly at certain worlds.*

Set-theoretic potentialism, in generality

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Countable transitive model potentialism has as its worlds all countable transitive models of ZFC, ordered by substructure.

- (Hamkins & Linnebo) The modal validities at any world for countable transitive model potentialism are bounded below by S4.2 and above by S5. Each bound is achieved exactly at certain worlds.

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S4.2 is S4 $+$ (.2).

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Let's see a very different flavor of example.

Extending models of set theory

Let $M \subseteq N$ be models of set theory.

- N is an end-extension of M if $b \in M$ and $N \models a \in b$ implies $a \in M$. That is, N doesn't add new elements to objects in M.
- N is moreover a rank-extension of M if $b \in N \setminus M$ implies rank $b \in N \setminus M$. That is, new elements are only added on top.

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What's new is M and N are not required to be well-founded. These notions make sense for ill-founded models.

- End-extensional potentialism has as worlds the countable models of set theory, ordered by end-extension.
- Rank-extensional potentialism has as worlds the countable models of set theory, ordered by rank-extension.

These are analogous to countable transitive model potentialism and Zermelian potentialism, but without a requirement that all worlds adhere to an external standard of well-foundedness.

Remark: If N end-extends well-founded M then any ill-foundedness in N must occur above the ordinals of M.

Set-theoretic potentialism allowing ill-founded worlds

Theorem (Hamkins & Woodin)

The modal validities at every world in rank-extensional potentialism are exactly S4.

Theorem (Hamkins & W.)

The modal validities at every world in end-extensional potentialism are exactly S4.

Remark: Trivially, S4 is a lower bound for any potentialist system—(S) expresses that the accessibility relation is reflexive and (4) expresses it is transitive.

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Both proofs follow a similar strategy: show that the potentialist system admits a universal finite sequence, a uniform definition for a finite sequence that can be freely extended by moving to the right larger world.

A universal finite sequence gives control statements witnessing that S4 is an upper bound, using that the class of finite pre-trees is complete for S4.

Better understanding one's commitments

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- If a potentalist system validates exactly S4, extensions come with choices of permanent consequence.
- The failure of (.2) means things that are possibly necessary can instead be made impossible.

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- For the rank- and end-extensional potentialist systems: as we extend and add new entries to the universal finite sequence these decisions are permanent.

Better understanding one's commitments

Taken together, these results show that the structure of modal truths for set theoretic potentialism depends upon whether there is a common standard of well-foundedness to which all worlds adhere.

Whether your Zermelo-style upwardly dynamic view of set allows such depends upon whether you think there is a definite notion of well-foundedness outside individual worlds.

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Enough for potentialism about sets, let's talk about potentialism for classes

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- But Zermelo isn't the only one to have an answer for this. Many mathematicians and philosophers have given answers to this question.

I want to focus on the species of answer which takes as a starting point a single, fixed universe of sets.

A popular—but insufficient—answer

Classes don't actually exist; talk of classes is just convenient shorthand for talk about (first-order) definable properties of sets.

- For example, "$\xi \in \mathrm{Ord}$" is shorthand for "ξ is transitive + linearly ordered by $\in$".

It's known that much *prima facie* talk about classes can be interpreted as only quantifying over sets—inner models, elementary embeddings of the universe, etc.

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The trouble is, there are uses of classes that cannot be captured just by looking at what is first-order definable.

Let's see two examples.

Kunen's inconsistency theorem

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If the only classes are the definable classes, this is a triviality:

- If j is definable without parameters, then so is the critical point of j, the least ordinal moved by j. But any elementary embedding $V \to V$ must fix every definable object, so $j(\operatorname{crit} j) = \operatorname{crit} j$. $\lightning$
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- (A small extra argument then yields that we also cannot have such j definable with parameters.)

If we think, as set theorists as a whole do, that there is substantive content to Kunen's theorem, it is in showing such j cannot even be an undefinable class.

Class forcing

- (Stanley, Friedman) If a class forcing notion $\mathbb{P}$ is pretame, then its forcing relations are definable.
- (Holy, Krapf, Lücke, Njegomir, Schlicht) But if $\mathbb{P}$ is not pretame then its forcing relations cannot be definable, even if we restrict to just the atomic formulae.
- (Gitman, Hamkins, Holy, Schlicht, W.) Indeed, we can exactly characterize a principle of class theory which is equivalent to the forcing theorem for every class forcing, namely the principle of Elementary Transfinite Recursion for recursions of height $\leq \mathrm{Ord}$. In particular, the class forcing theorem is equivalent to a principle asserting the existence of certain kinds of truth predicates.

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If we want to be able to talk about class forcing in full generality we need undefinable classes.

What are classes then?

- Philosophers of mathematics and mathematicians have proposed different answers to what classes are, and how they differ from sets.
- Some of them admit a natural potentialist reading.
- (Barton & W.) Studying the mathematics of potentialism for sets can help us to better understand our commitments for what sets are. Perhaps the same can be done with potentialism for classes.

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Let's survey a couple of these answers for what classes are.

Fujimoto's liberal predicativism

Developed by Fujimoto, following earlier work by Parsons.

Quote (Fujimoto 2019)

Our proposal is to interpret the [class] quantifier $\exists X$ as "there exists an admissible predicate such that..." or "there is a predicate *we may admissibly introduce* such that..." (emphasis mine)

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- Classes are distinct from sets because they are part of language—predicates—unlike sets.
- But this goes beyond just definable classes. In particular, Fujimoto explicitly allows truth predicates as admissible predicates.
- Indeed, he explicitly motivates his project with the need to allow talk of undefinable classes.

Linnebo's individuation of properties

Linnebo proposes a theory of properties on which properties are successively individuated along the ordinals.

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- Linnebo's properties are not classes, since they are intensional objects. But we can interpret class theory by looking at extensions of properties.
- There is a hierarchy of classes, based on 'when' a property giving that class is first individuated.

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- Both take the sets as a fixed totality.
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- Less clear from what I quoted, but: In both views, iterated truth predicates play an important role in measuring what classes exist.

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- Both can be seen as building up the classes by allowing more and more.
- This has a natural interpretation in a modal framework.
- Less clear from what I quoted, but: In both views, iterated truth predicates play an important role in measuring what classes exist.

There is a Tarski-style hierarchy to truth predicates: truth about V, truth about truth, truth about truth about truth, and so on.

Iterated truth predicates are a device to put this hierarchy into a single class. Each iterated truth predicate has a length, which may be transfinite, and possibly even of length $> \mathrm{Ord}$.

Truth potentialist systems

Fix countable $M \models \text{ZFC}$, to be the sets of the worlds. A truth potentialist system over M has worlds $(M, \mathcal{X})$ with classes $\mathcal{X}$ over M:

- Each world $(M, \mathcal{X})$ satisfies predicative comprehension and class replacement.
- The definable classes of M form a world.
- If $(M, \mathcal{X})$ is a world and $A \in \mathcal{X}$, then there is a larger world containing the truth predicate relative to A as a parameter.

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- Modification: require there be larger worlds with iterated truth predicates relative to A of any length which exists. (This gives a modal version of Fujimoto's appraoch.)
- More modification: require there be larger worlds with iterated truth predicates of length bounded by some Λ. (E.g. $\Lambda = \mathrm{Ord}$ corresponds to Linnebo's approach.)

Basic facts about truth potentialist systems

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- The worlds are the classes definable from the length ξ iterated truth predicate (relative to no parameter), for different finite ξ.
- Ditto for requiring iterated truth predicates of bounded transfinite length. (Just need to allow longer lengths ξ, less than the bound.)
- And for requiring iterated truth predicates without bounds on their length. (You seem to need an extra technical condition here, about the worlds being correct about which classes are well-founded.)

The modal logic of truth potentialism

Theorem (Barton & W.)

Fix the sets M.

- *The smallest truth potentialist system for M validates* S4.3, *and ditto for the transfinite versions.*
- *If the lengths are unbounded or if the bound Λ is closed under addition $<\omega^2$, then the modal validities are exactly* S4.3.

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Proof sketch: For the first: observe this potentialist system is linearly ordered, similar to the Zermelian case.

For the second: we need a long ratchet.

β_ξ: "the length ξ iterated truth predicate (relative to no parameter) exists".

Then $\langle \beta_\xi : \xi < \Lambda \rangle$ is a long ratchet. The point is, to make Leibman's lemma (long ratchets give S4.3 as an upper bound) work, we only need that the length of the ratchet is closed under addition $<\omega^2$.

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- Indeed, even class theories with full impredicative comprehension don't suffice to guarantee a global well-order exists.
- A global well-order can be added by a class forcing which doesn't add sets.
  The natural way: conditions are set-sized partial well-orders of the universe.
  This is forcing-equivalent to adding a Cohen-generic class of ordinals.

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Truth potentialism with a global well-order

A class potentialist may want to say there's a (first-order) definable global well-order, but this has substantial cost. (It's equivalent to requiring that the sets satisfy $\exists x\ V = \mathrm{HOD}(\{x\})$.)

An alternative: instead of starting with a base world of the definable classes, start with a base world which contains a (possibly non-definable) global well-order. That is, the base world consists of all classes (first-order) definable from the global well-order.

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Maybe there's yet another alternative?

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Yet another approach to allowing global well-orders

We keep the base world as just the definable classes, but we add a new extension rule, saying we can extend to a larger world with a global well-order.

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The trouble is, this can be even more destructive!

Lemma (Killing Truth, W.)

Let M be a countable, ω-standard model of ZFC*. Then there is a Cohen-generic class C of ordinals so that C and the truth predicate for M cannot both be in the same* NBG*-expansion for M. Indeed, from C and the truth predicate you can defined a cofinal ω-sequence in the ordinals of M.*

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This gives a rather nasty failure of the (.2) axiom: "the truth predicate for the sets exists" is possibly necessary, but in the extension by C it is impossible.

Indeed, this lemma has consequences for other potentialist systems:

- The potentialist system consisting of all NBG-expansions of M does not validate (.2).

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- Accept that the modal structure of the multiverse is very different than the orderly structure of vanilla truth potentialism.
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- Put strong restrictions on the first-order theory of the sets.
- Accept that the modal structure of the multiverse is very different than the orderly structure of vanilla truth potentialism.
- Explicate a third option for global well-orders besides being definable or being generic.
- Some other option I'm not clever enough to recognize.

A conjecture, and future work

Let T be a reasonable class theory, such as NBG or KM and fix a countable model M of ZFC.

Conjecture

The potentialist system consisting of all T-expansions of M has exactly S4 *as its modal validities.*

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Let T be a reasonable class theory, such as NBG or KM and fix a countable model M of ZFC.

Conjecture

The potentialist system consisting of all T-expansions of M has exactly S4 *as its modal validities.*

Some evidence:

- The killing truth lemma implies S4.2 is too strong an upper bound for weak enough T.
- The analogous fact is true in second-order arithmetic, with full impredicative comprehension for 'classes'. (This is a corollary of the Hamkins & W. result about end-extensional potentialism.)
- For very strong T and ω-nonstandard M I can prove this. (But that is the least interesting instance of this conjecture...)

Thank you for listening!

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