Coding sets into inner mantles

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Set theoretic geology

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- Geology can be done in ZFC. (We seem to need AC; Gitman–Johnstone and Usuba have partial results.)
- All worlds in the generic multiverse are at most two steps away: M is a forcing extension of a ground of N.

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The bedrock axiom is true in, e.g., L while it is destroyed by set forcing.

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- The bedrock axiom $\mathrm{V} = \mathrm{M}$ asserts there are no nontrivial grounds.
- (Reitz) You can class force the bedrock axiom.

Do a set-support iteration of lottery sums to generically make the GCH fail/succeed at each regular cardinal.

Non-absoluteness and the mantle

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- ($\mathrm{V} \supseteq \mathrm{M}^{\mathrm{V}[G]}$) Consider $x \in \mathrm{V}[G] \setminus \mathrm{V}$. The forcing $\mathbb{P}$ is a progressively distributive product, so we can factor it into the product of a set-sized head and a sufficiently distributive tail so that the tail forcing could not add x. But then $\mathrm{V}[G^{\mathsf{tail}}]$ is a ground which misses x.

Inner Mantles

The sequence of inner mantles M^i is defined inductively.

- $\mathrm{M}^0 = \mathrm{V}$;
- $\mathrm{M}^{i+1} = \mathrm{M}^{\mathrm{M}^i}$;
- $\mathrm{M}^\ell = \bigcap_{i<\ell} \mathrm{M}^i$ for limit ordinals ℓ.

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Do an $\eta^\star$ iteration of the FHR forcing.

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- **Question** (Fuchs–Hamkins–Reitz): Can this fail at limit stages? More precisely:
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- Must M^ℓ satisfy AC, if M^i is definable for all $i < \ell$?
- Compare to the classical questions about iterated HOD, answered by Harrington and McAloon.

Set theoretic arbology

Boise is the City of Trees, so I'm obligated to use trees in this talk.

To answer the FHR questions, we need to precisely control which sets get into which inner mantles. For this we will use what I call tree iterations.

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- Coding x into the Cohen pattern once gets it into HOD, but isn't enough to get it into M.
- Instead, code it Ord often: force with the set-support product of $\mathrm{Add}^{\mathrm{L}}(\aleph_{\omega\cdot\xi+n}, 1)$ for $\xi \in \mathrm{Ord}$ and $n \in x$.
- In $\mathrm{L}[x][\bar{c}]$: we can recover x in any ground by looking at the Cohen pattern in $[\aleph_{\omega\cdot\xi}, \aleph_{\omega\cdot\xi+\omega})$ for some large enough ξ. So $x \in \mathrm{M}$.

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To get x into M^2 we'd want to in turn code each $c_{\omega\cdot\xi+n}$ into the Cohen pattern cofinally often, and so on to get even deeper.

So there's a tree-like structure to the order of the coding.

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An aside: when you need Ord much space

Hilbert's Ord-Hotel

- To code a set into the mantle, we need Ord much space.
- So if we're coding multiple sets into mantles, we need multiple Ord-sized regions for coding.
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- So if we're coding multiple sets into mantles, we need multiple Ord-sized regions for coding.
- In a region R: code whether $i \in x$ by whether the i-th cardinal in R contains a Cohen subset.
- This is easily arranged, and if our forcings preserve cardinals then it is easy to do so in an absolute way.

Tree iterations

Rather than do a linear iteration, we want to do an iteration $\mathbb{P}$ along a tree T.

- For convenience, will always do trivial forcing at the root stage.
- The generic at stage $s \in T$ should be generic over $\mathrm{V}[G \upharpoonright {<}s]$
- If $s_0 \neq s_1 \in T$ have infimum t, then the generics at stage s_0 and s_1 should be mutually generic over $\mathrm{V}[G \upharpoonright {\leq}t]$.

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For my context:

- All trees are well-founded.
- All supports are set-support.

Technical lemmata

Non-linear iterations have been studied before, e.g. by Groszek and Jech (1991). Specializing some of their work to my context, we can get:

- **Safety Lemma**: A tree iteration of Cohen coding forcings along a tree T only adds a Cohen subset to α if some iterand $\dot{\mathbb{Q}}_s$ for a stage $s \in T$ adds a Cohen to α, and this iterand is the only thing adding a Cohen.
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Let $T_{-1} \subseteq T$ be the subtree consisting of all non-leaf nodes. Using Reitz's technology of generalized Cohen iterations can see:

- A tree iteration $\mathbb{P}$ along T of Cohen coding forcings can be factored as $(\mathbb{P} \upharpoonright T_{-1}) * \dot{\mathbb{R}}$ where $\dot{\mathbb{R}}$ is a progressively distributive product.

AC may consistently fail at the ω-th inner mantle

Theorem (W.)

There is a class forcing extension of L in which the ω-th inner mantle M^ω is a definable inner model of ZF in which $\mathcal{P}(\omega)$ cannot be well-ordered.

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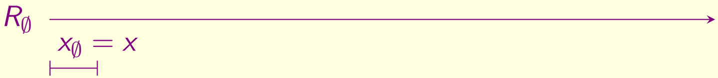

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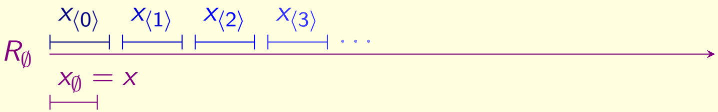

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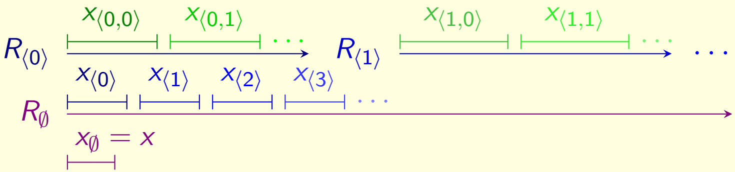

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- Now code each code Ord often.

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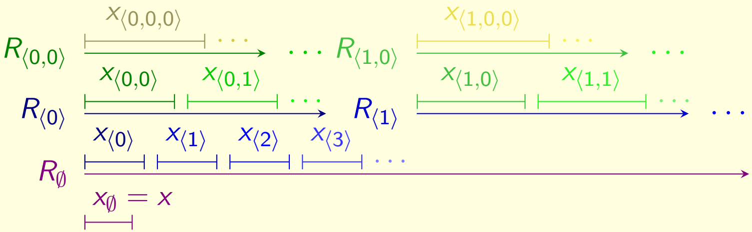

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- Now code each code Ord often.
- Repeat with a stage for each $s \in {}^{\leq k}\mathrm{Ord}$.

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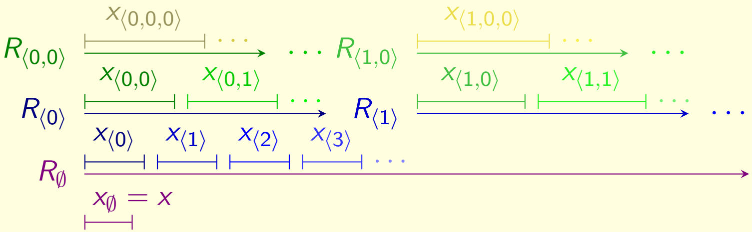

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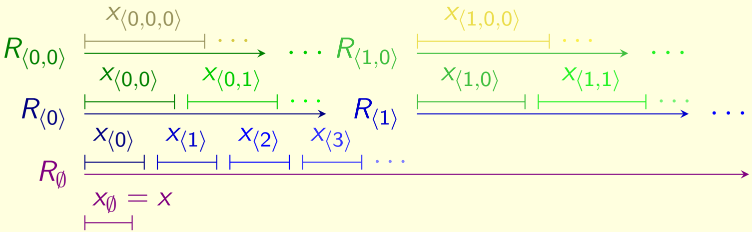

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Call this tree-like coding by $\mathbb{T}_k(x)$ for short.

$k =$ the height

$x =$ the set to code

The coding forcing: putting it together

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Lemma: Taking inner mantles in $\mathrm{L}[A][\bar{c}]$ corresponds to climbing down the tree of generics:

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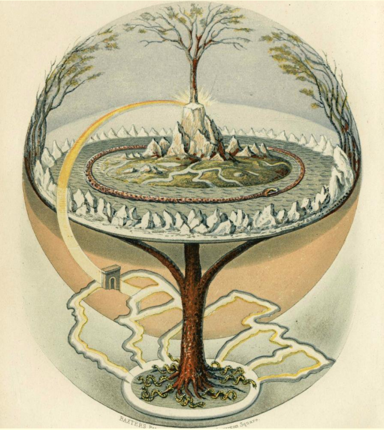
where $\bar{c}_{-k}$ is the subtree of generics from nodes distance at least k from the farthest leaf node above.



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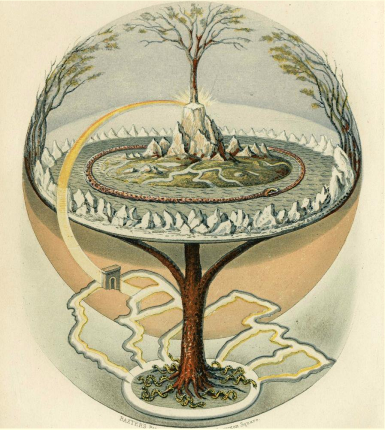

In particular, $A^k \in \mathrm{M}^k \setminus \mathrm{M}^{k+1}$

M^ω may fail to be a definable class

Theorem (W.)

There is a class-forcing extension of L *whose* M^ω *can define the truth predicate for* L.

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- Then no outer model of ZF can define the truth predicate for L, as else it could define a bijection $\omega \to \mathrm{Ord}$.
- So the class-forcing extension for the theorem cannot have M^ω as a definable class, as then it could define the truth predicate for L.

Aside: ω-nonstandard models

You can do this coding truth-in-L construction in ω-nonstandard models.

Corollary (W.)

- *There is ω-nonstandard $N \models \mathsf{ZFC}$ so that, in N, M^k is a definable class if and only if k is standard.*
- *For any ω-nonstandard $L \models \mathsf{ZFC} + \mathrm{V} = \mathrm{L}$ and any $e \in \omega^L$ there is a class forcing extension $L[G]$ in which M^k is a definable class if and only if $k < e + n$ for some standard n.*

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- Is there anything special about ω? Can the same results be obtained at any limit stage?

Thank you!

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