

Incompleteness and the universal algorithm

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A very accurate and nuanced early history of the foundations of computation



Find an algorithm to solve the
*Entscheidungsproblem**

* (Given a logical formula determine whether it is true in all structures.)



No.

In a bit more detail

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- Alonzo Church (1936), Alan Turing (1936), and others gave formalizations, which turn out to be equivalent.
- And since then there has been an explosion in equivalent characterizations, e.g. (an idealized version of) your favorite programming language.
- An advantage to giving a talk in 2020 is that computers are so ubiquitous I don't need to give you the formal definition of a **Turing machine** (TM).

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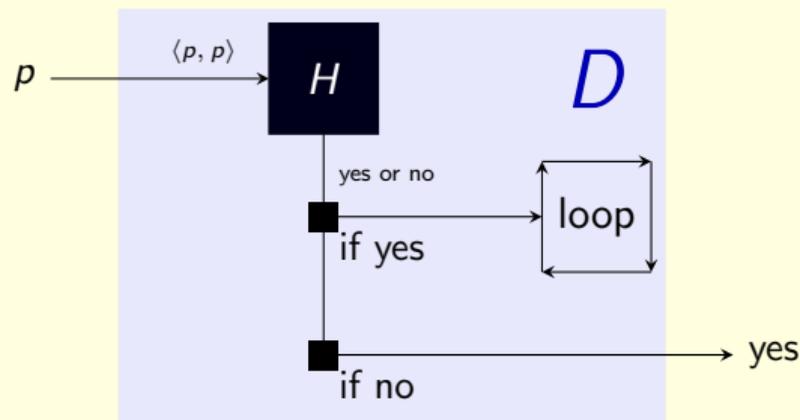
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- **Hard part!** Turing showed that TMs are powerful enough to do computations involving other TMs. Indeed, he showed there is a **universal machine** which can simulate any TM.
- **Other hard part!** Turing's conceptual analysis to argue that his formalization correctly captures the intuitive notion of computability.
- **Easy part!** Do a diagonalization argument.

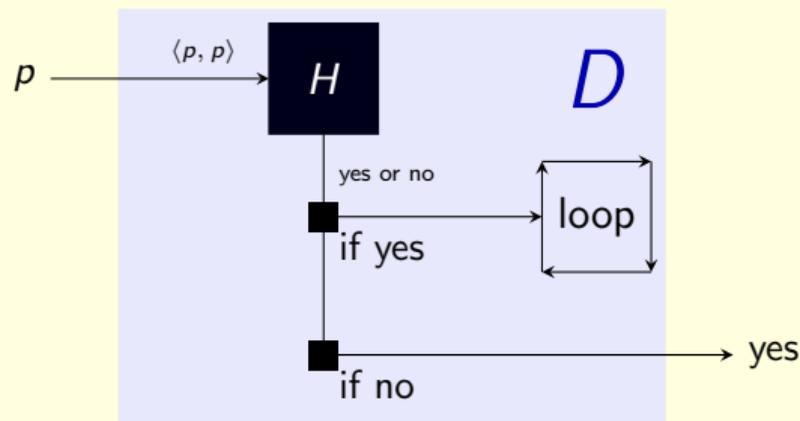
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Toward a contradiction suppose H is a TM which decides whether or not p halts on input n . Let's build a new TM D .



Now ask: what happens when D is input to D ?
Then it halts iff it doesn't. ⚡

Let's talk about another kind of undecidability, in terms of what you can prove instead of what you can compute.

And then we'll see how the two kinds of undecidability relate.

A very accurate and nuanced history of the incompleteness theorems



Find axioms that decide all questions of natural number arithmetic.



No.

The incompleteness theorems

Peano arithmetic (PA) axiomatizes natural number arithmetic: axioms of discretely ordered semirings + induction axioms.

Theorem (Gödel's first and second incompleteness theorems)

- 1 *No computably axiomatizable extension of PA is complete. There must be an arithmetic statement it neither proves nor disproves.*
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We need the restriction. True arithmetic TA—the set of all truths of \mathbb{N} —is a complete extension of PA. (Indeed, there are complete extensions of PA which are arithmetically definable.)

Arithmetization

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 - What is a finite semiring? It's a tuple $\langle R, +, \times \rangle$ satisfying certain axioms. Represent R by a sequence of its elements and $+$ and \times by sequences giving their multiplication tables. So you can write an arithmetic formula which expresses “ n codes a finite semiring”.

$+$	1	a	b	0	\times	1	a	b	0
1	1	1	1	1	1	1	a	b	0
a	1	a	1	a	a	a	a	0	0
b	1	1	b	b	b	b	0	b	0
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- More relevant to this talk, objects like Turing machines or logical formulae can be coded in arithmetic.
- Statements like “PA does not prove $0 = 1$ ” or “such and such Turing machine halts” can be cast as statements in arithmetic.

Arithmetization

0 substituted for $[s]_0$, and φ with $[s]_0 + 1$ substituted for $[s]_0$. Now for the gory details. We define the relation $\text{PA}(x)$, expressing that x is the Gödel-number of a Peano axiom by the formula

$$x = n_1 \vee \dots \vee x = n_{15} \vee \left(\begin{array}{l} \exists y, s \subseteq x \exists n \leq s \left(\begin{array}{l} \text{Form}(y) \wedge \text{len}(s) = n \wedge \\ \forall i < \text{len}(n) \text{Free}(y, [s]_i) \wedge \forall j \leq y (\text{Free}(y, j) \rightarrow \exists k \leq s [s]_k = j) \wedge \\ \exists t \subseteq s \exists u, w \left(\begin{array}{l} \text{len}(t) = \text{len}(s) - 1 \wedge \forall i < \text{len}(t) [t]_i = [s]_{i+1} \wedge \\ u = \text{Sub}(y, [s]_0, \ulcorner 0 \urcorner) \wedge w = \text{Sub}(y, [s]_0, \ulcorner [s]_0 + 1 \urcorner) \wedge \\ x = \ulcorner (\forall t (u \wedge (\forall [s]_0 (y \rightarrow w) \rightarrow \forall [s]_0 y))) \urcorner \end{array} \right) \end{array} \right) \end{array} \right).$$

(Taken with permission from Victoria Gitman's lecture notes for Mathematical Logic, Spring 2013.)

Self-reference

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Formally: for any formula $\varphi(x)$ there's a sentence σ so that σ is PA-provably equivalent to $\varphi(\sigma)$.

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Formally: for any formula $\varphi(x)$ there's a sentence σ so that σ is PA-provably equivalent to $\varphi(\sigma)$.
- You can now prove a form of the first incompleteness theorem by considering σ PA-provably equivalent to “ σ is not PA-provable”.
- Suppose PA proves σ . Then PA proves that σ is not provable. Whence PA does not prove σ . $\not\vdash$

Suppose PA proves $\neg\sigma$. Then PA proves that σ is provable. Whence PA does prove σ . $\not\vdash$

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Some say the incompleteness theorems are difficult to prove. But if you handwave over the actual hard parts you can fit the proof on one slide. :)

Self-reference

- Also have self-reference for computability theory, via the **Kleene recursion theorem**. Informally, Turing machines can refer to themselves.
- Formally: for any computable partial function $F(x, y)$ there's a TM p so that p computes the function $y \mapsto F(p, y)$.

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A fun application: programming languages admit **quines**—programs that output their own source code.

```
;; Quine in Common Lisp
((lambda (x) (list x (list 'quote x))))
'(lambda (x) (list x (list 'quote x))))
```

Incompleteness and Turing machines

The incompleteness theorems can be recast as saying that whether certain Turing machines halt is undecidable.

A TM p :

- Look at all length 1 proofs from the first 1 axiom of PA.
- Then look at all length 2 proofs from the first 2 axioms of PA.
- \vdots
- If at any point you see a proof that ends with $0 = 1$, halt and output affirmatively.

Whether p halts is independent of PA.

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- Adam Yedidia and Scott Aaronson do even better.
- They constructed a TM of size 7910 so that whether it halts is independent of ZFC, but ZFC + large cardinals does prove it halts.

(Specifically an ineffable cardinal will do.)

If you liked Gödel's incompleteness theorems, you'll love his completeness theorem

Theorem (Gödel's Completeness Theorem)

- 1 *A set of axioms T is consistent if and only if there is a structure satisfying T .*
- 2 *φ is true in every structure satisfying T if and only if φ is a theorem of T .*

(This is for axioms in first-order logic.)

- This lets us translate talk about proofs, consistency, etc. to talk about structures.
- The incompleteness theorems plus the completeness theorem together imply there must be non-isomorphic structures satisfying the axioms of arithmetic.

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What could these even look like???

Nonstandard models of arithmetic

A model of (Peano) arithmetic is a discretely ordered semiring whose definable subsets are inductive.

M 

- $X \subseteq M$ is **definable** if you can express $x \in X$ just by quantifying over the elements of M and using the semiring operations and order of M .
- $X \subseteq M$ is **inductive** if $0 \in X$ and $a \in X \Rightarrow a + 1 \in X$ implies $X = M$.

Nonstandard models of arithmetic

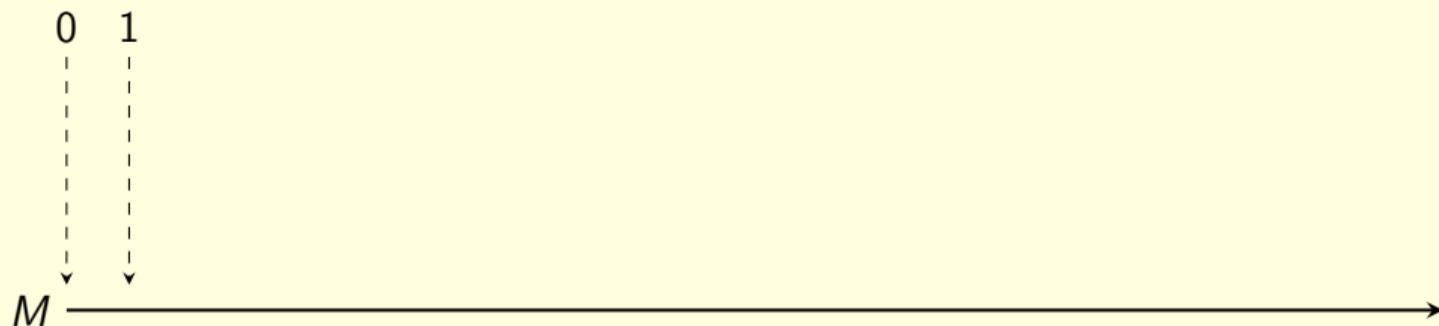
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M has a least element 0^M (= the additive identity for M) because the set $\{x \in M : x \geq 0^M\}$ satisfies the inductive hypotheses.

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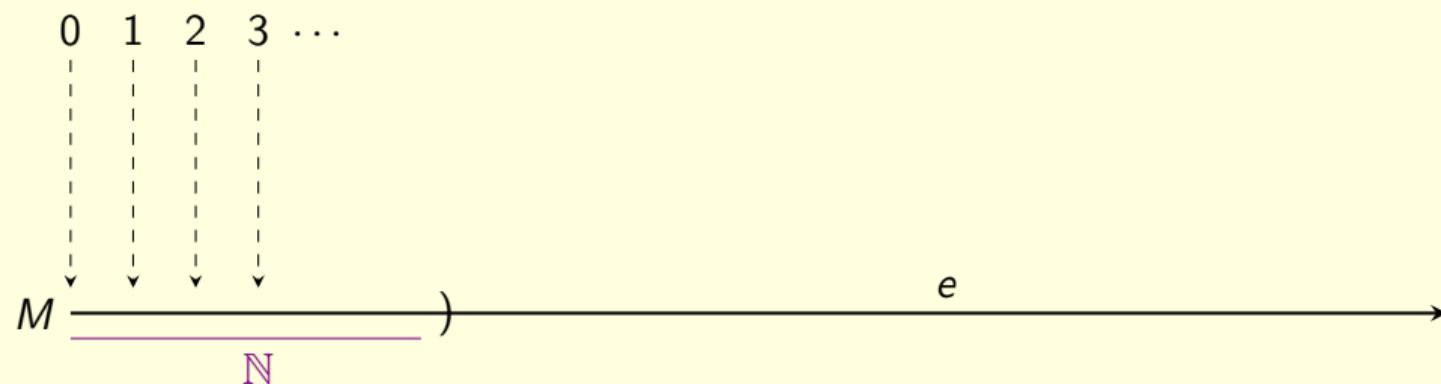
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\mathbb{N} embeds as an initial segment on any model of arithmetic.

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If $e \in M \setminus \mathbb{N}$ then $e > n$ for all $n \in \mathbb{N}$.

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All non-zero elements have a predecessor because

$$\{0\} \cup \{a \in M : a \text{ has a predecessor}\}$$

satisfies the induction hypotheses.

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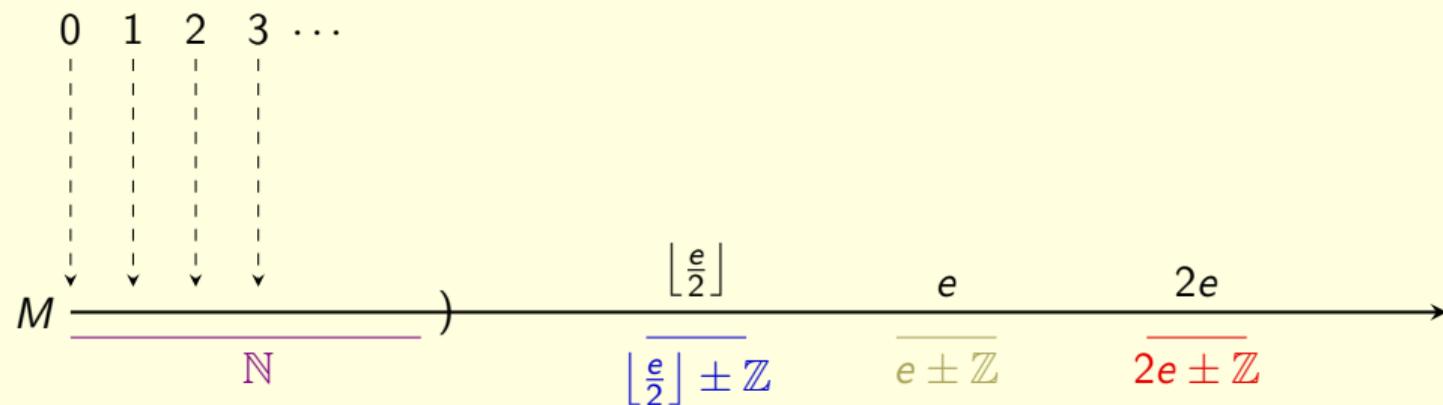
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$e + n < e + e = 2e$ for all $n \in \mathbb{N}$.

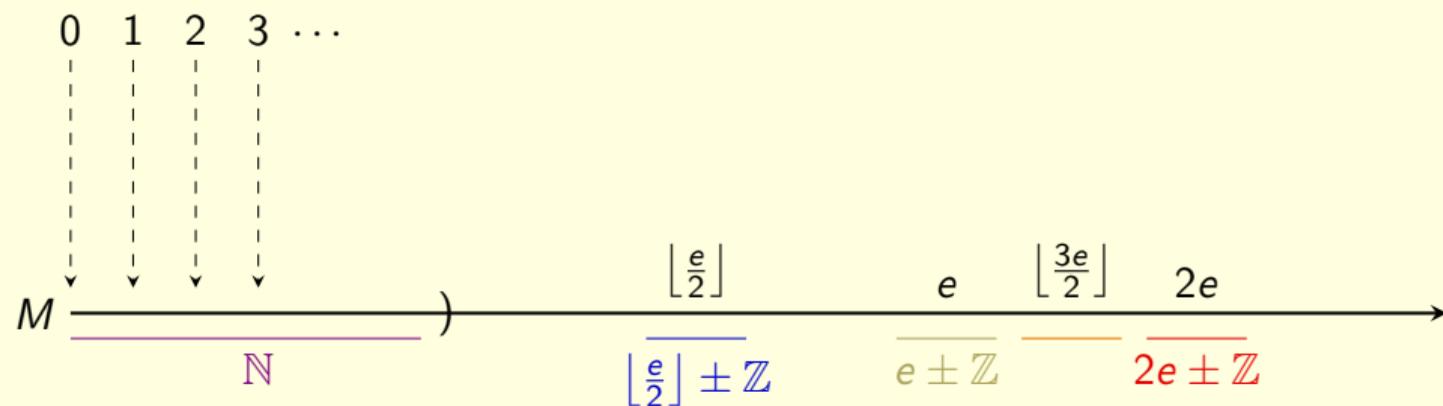
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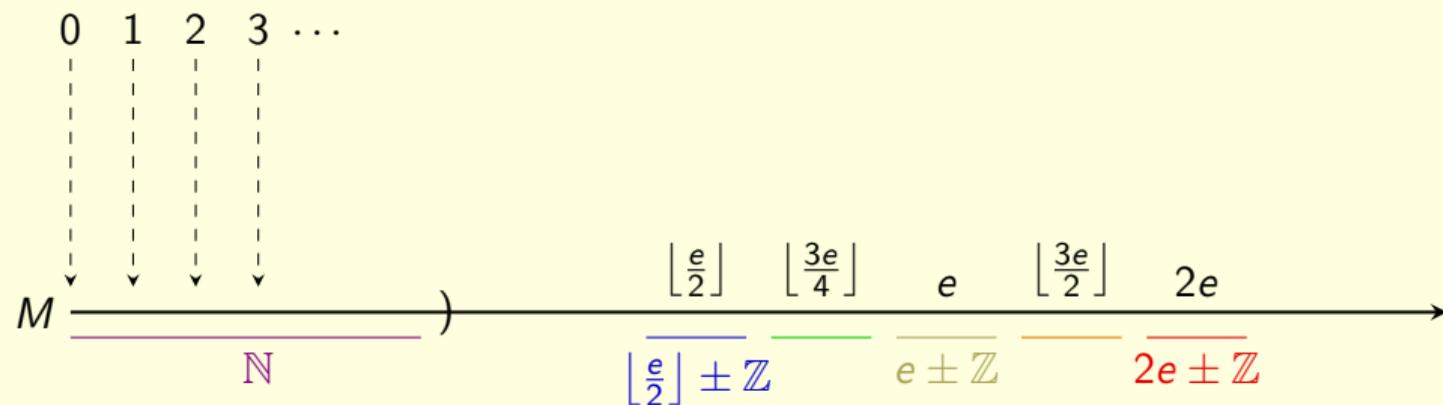
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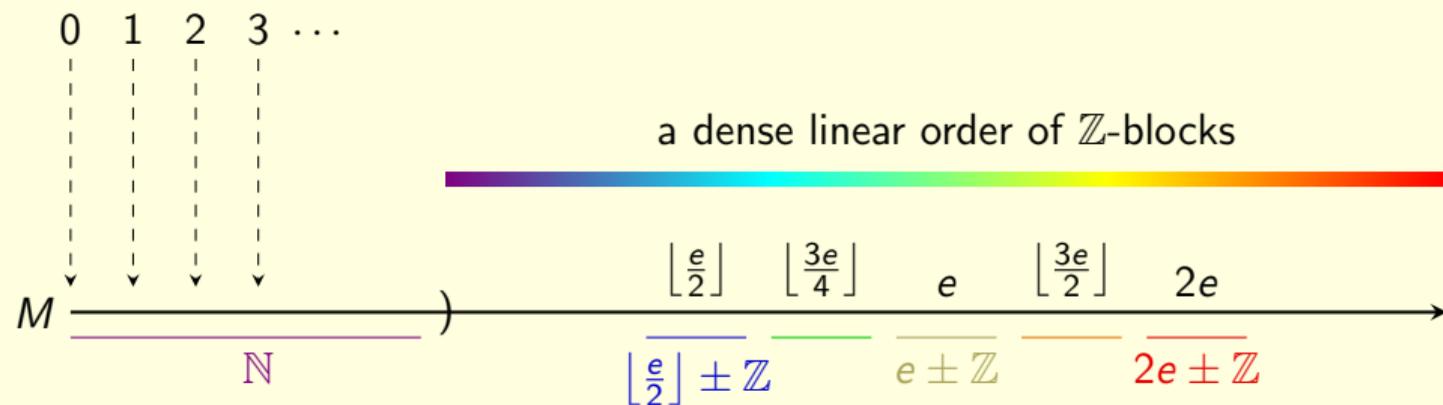
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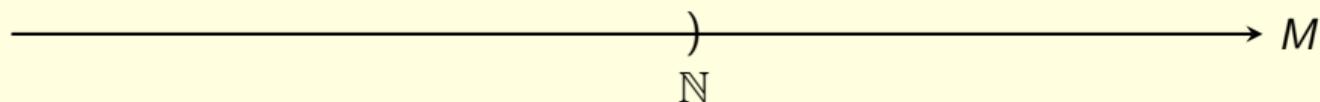
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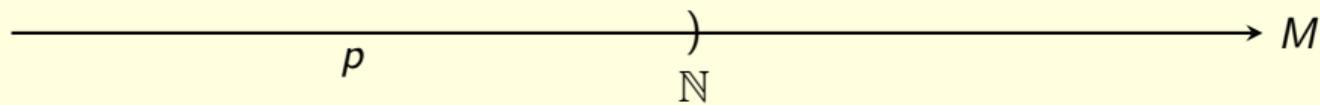
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- **Open Question** (Harvey Friedman): \mathbb{N} has the property that if a model of arithmetic is order-isomorphic to it then they are fully isomorphic. Does any other model of arithmetic have this property?
- (Stanley Tennenbaum) If M is nonstandard then neither the $+$ nor \times of M is a computable function.

Turing machines in a nonstandard world



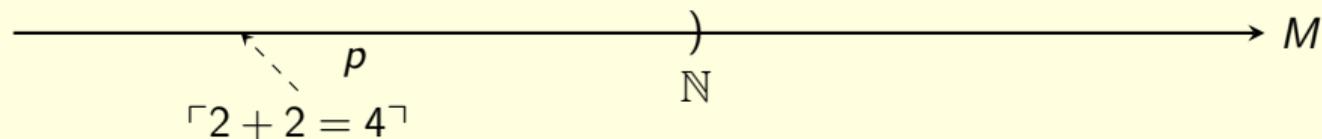
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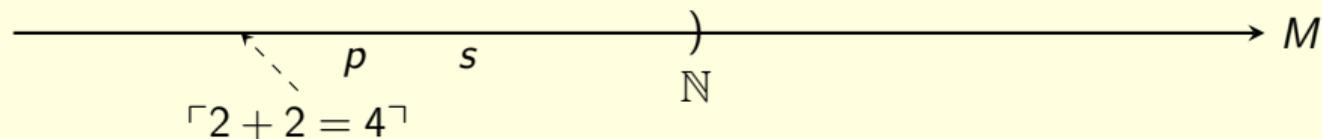
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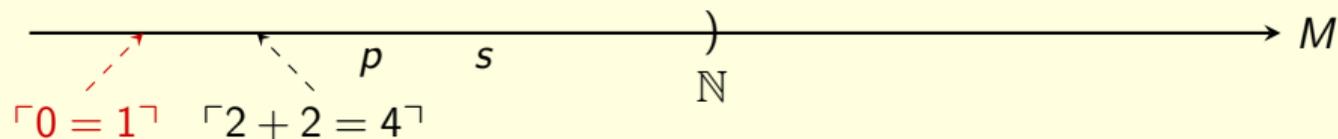
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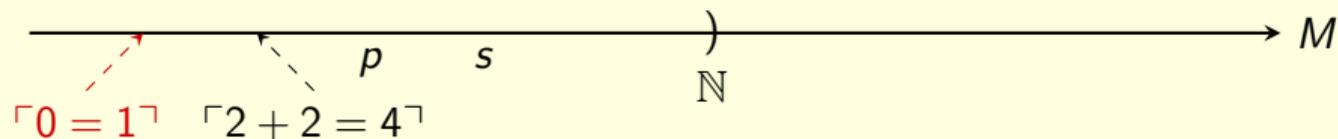
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(s is a number coding the sequence of computation steps. Checking that s has this property only requires looking in \mathbb{N} .)

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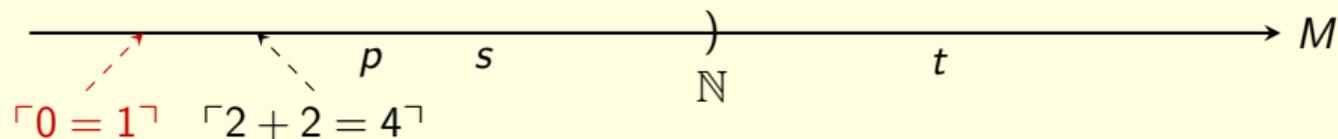
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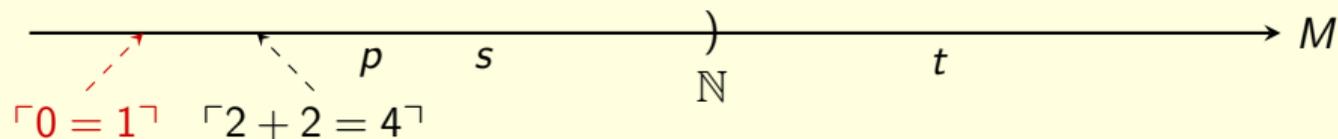
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- But what if we run p in nonstandard M which thinks arithmetic is inconsistent?
- Then there is a computation log t witnessing that p outputs $\ulcorner 0 = 1 \urcorner$. But t must be nonstandard! In other words, we had to run p for a nonstandard number of steps to output $\ulcorner 0 = 1 \urcorner$.

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- Then there is a computation log t witnessing that p outputs $\lceil 0 = 1 \rceil$. But t must be nonstandard! In other words, we had to run p for a nonstandard number of steps to output $\lceil 0 = 1 \rceil$.
- **The point:** By moving to a larger world we made p output more numbers.

The absoluteness of computability

In summary:

- The statement “the TM p outputs n for some input” is **upward absolute**—if it’s true it stays true if we **end-extend** to a larger model.

(Logicians call this sort of statement a Σ_1 statement. By the MRDP theorem, these are the statements equivalent to one whose only quantifiers are a block of \exists s.)

- But the statement “the TM p does **not** output n for some input” is not upward absolute. (It is **downward absolute** though.)

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By Gödel’s completeness theorem plus the last slide, Peano arithmetic proves every true (i.e. in \mathbb{N}) statement of this form.

- But the statement “the TM p does **not** output n for some input” is not upward absolute. (It is **downward absolute** though.)

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Both the first and second incompleteness theorems are about statements of this form.

Woodin's universal algorithm

We've seen that the behavior of a Turing machine can be undecidable.

- **Proof theoretic:** It may be independent of PA how p behaves.
- **Model theoretic:** Running p in different nonstandard models of arithmetic may produce different behavior.

I want to talk about a striking case of the undecidability of how Turing machines behave, due to W. Hugh Woodin, where p can output anything at all if run in the right universe!

DISTINGUISHED LECTURE SERIES

NUS National University of Singapore IMS Institute for Mathematical Sciences

W. HUGH WOODIN

W. Hugh Woodin is Professor of Philosophy and of Mathematics at Harvard University. Woodin has been an ICM speaker three times, is a member of the American Academy of Arts and Sciences, and for nearly 20 years has been a Distinguished Visiting Professor in the Mathematics Department at NUS.

Lecture 1: A new basis theorem for Σ_1^1 sets
Thursday, 6 June 2019, 9:30-10:30am

This is the first lecture in a two-part series on recent applications of the fine-structure of inner models to problems in descriptive set theory. The focus of this first lecture will be on the projective sets and simple generalizations. The context will be determinacy hypotheses.

Lecture 2: Counting Woodin cardinals in HOD
Monday, 10 June 2019, 9:30-10:30am

The final synthesis of fine-structure and determinacy will yield a number of theorems about HOD in the context of the Axiom of Determinacy. However, there are some of these expected theorems which can be proved now before that final synthesis is achieved. We focus on one such recent theorem which concerns the relationship between the number of Woodin cardinals in HOD and the descriptive set theory of the universe within which HOD is defined.

One application shows that the axiom $V = Ultimate L$ implies the \diamond Conjecture.

Host:
Auditorium
Institute for Mathematical Sciences
3 Prince George's Park, Singapore 118402

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Theorem (Woodin)

There is a Turing machine p with the following properties.

- ① *p provably enumerates a finite sequence.*
- ② *Running p inside \mathbb{N} never produces any output, i.e. it enumerates the empty sequence.*
- ③ *But, for any finite sequence s of natural numbers there is a nonstandard model of arithmetic M so that running p in M enumerates exactly s .*

Woodin's algorithm

(This construction for Woodin's theorem is due to Joel David Hamkins.)

The Turing machine p :

- p searches through the proofs of Peano arithmetic, looking at the theorems they prove.
- p is looking for a theorem of the form “ p does **not** enumerate the sequence s ”, for s some nonempty finite sequence of numbers.
(p can refer to itself by the Kleene Recursion theorem.)
- If p ever sees this, then p outputs the sequence s .

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Claim: Run in \mathbb{N} , p outputs the empty sequence.

- Otherwise p outputs some s when ran in \mathbb{N} .
- Then, Peano arithmetic proves the true Σ_1 statement “ p outputs s ”.
- But by the definition of p , this also means that Peano arithmetic proves that p does not output s .
- This would mean that Peano arithmetic is inconsistent. But it's not.

Checking the extension property

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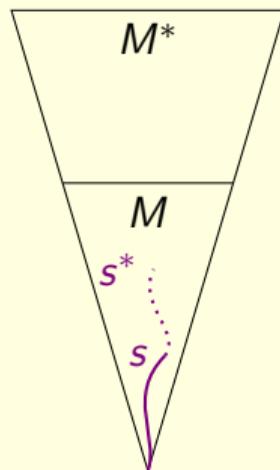
- Otherwise “ p does not output s ” is a theorem of Peano arithmetic.
- But then running p in \mathbb{N} would output a nonempty sequence. We just saw that is not the case.
- So by Gödel’s completeness theorem we can find a model of arithmetic in which p outputs s . □

Woodin's universal algorithm, general form

Theorem (Woodin)

There is a Turing machine p with the following properties.

- 1 p provably enumerates a finite sequence.
- 2 Running p inside \mathbb{N} never produces any output, i.e. it enumerates the empty sequence.
- 3 Suppose M a model of arithmetic in which p enumerates s and that s^* is a sequence in M which extends s . Then we can end-extend M to a larger model of arithmetic M^* in which p enumerates s^* .

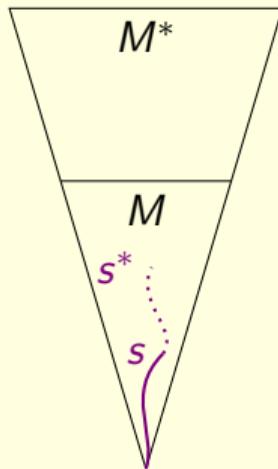


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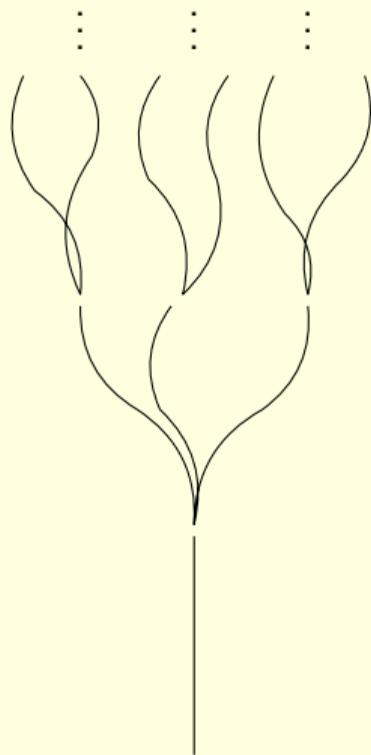
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Proof idea: Tweak the definition of p so it works in stages. Then do a similar argument, but internally to M . Need some more technical lemmata to check that the argument can be arithmetized.

Arithmetic potentialism



- Imagine climbing through the tree of nonstandard models of arithmetic, continually end-extending.
- This **potentialist system** gives a nonstandard twist on Aristotle's notion of the **potential infinite**.
- There is a natural interpretation in **modal logic**—extend ordinary logic by adding two new operators
 - $\Box\varphi$ means φ is **necessarily true**—true in all extensions.
 - $\Diamond\varphi$ means φ is **possibly true**—true in some extension.
- (Hamkins) Can use Woodin's universal algorithm to calculate which modal assertions are valid (true in any world under any substitution of variables).
(Namely, those in the modal theory $S4$.)
- (Hamkins) There are models of arithmetic which satisfy the **maximality principle**—if $\Diamond\Box\varphi$ then φ .

Woodin's ideas have been built upon to find universal “algorithms” in a set theoretic setting.

- (Hamkins & Woodin) There is a universal “algorithm” for **rank-extensions** of models of set theory.
- (Hamkins & Williams) There is a universal “algorithm” for **end-extensions** of models of set theory.
- (Hamkins & Williams) There is a universal “algorithm” for **ω -models** of second-order arithmetic.

Thank you!



Turing surrounded by a multiverse of models of arithmetic in full bloom

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