Incompleteness, the universal algorithm, and arithmetic potentialism

Kameryn J. Williams

University of Hawaiʻi at Mānoa

Talk Math With Your Friends
2020 September 17

A very accurate and nuanced early history of the foundations of computation

Find an algorithm to solve the *Entscheidungsproblem**.

No.

* (Given a logical formula determine whether it is true in all structures.)

In a bit more detail

- The strategy to show an algorithm solves the *Entscheidungsproblem* is straightforward: exhibit the algorithm and check it does what you want.
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- And since then there has been an explosion in equivalent characterizations, e.g. (an idealized version of) your favorite programming language.

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- And since then there has been an explosion in equivalent characterizations, e.g. (an idealized version of) your favorite programming language.
- An advantage to giving a talk in 2020 is that computers are so ubiquitous I don't need to give you the formal definition of a Turing machine (TM).

Turing reduced the *Entscheidungsproblem* to the halting problem

Theorem (Turing)

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- Easy part! Do a diagonalization argument.

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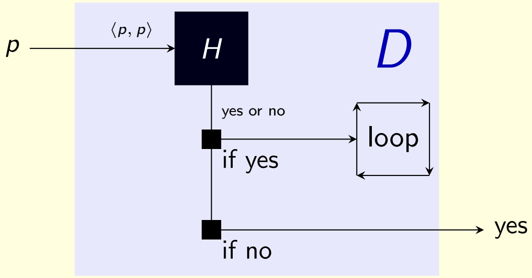
- Other hard part! Turing's conceptual analysis to argue that his formalization correctly captures the intuitive notion of computability.
- Easy part! Do a diagonalization argument.

The easy part: the diagonalization argument

Toward a contradiction suppose H is a TM which decides whether or not p halts on input n. Let's build a new TM D.



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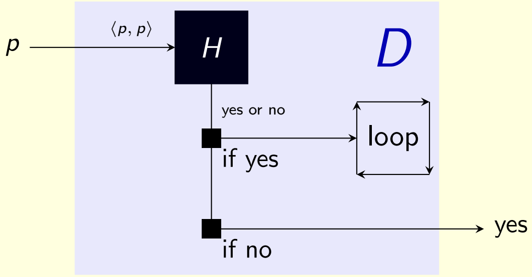

Now ask: what happens when D is input to D?
Then it halts iff it doesn't. ↯

From computability theory to proof theory

Let's talk about another kind of undecidability, in terms of what you can prove instead of what you can compute. And then we'll see how the two kinds of undecidability relate.

A very accurate and nuanced history of the incompleteness theorems

Find axioms that decide all questions of natural number arithmetic.

No.

The incompleteness theorems

Peano arithmetic (PA) axiomatizes natural number arithmetic: axioms of discretely ordered semirings + induction axioms.

Theorem (Gödel's first and second incompleteness theorems)

1. *No computably axiomatizable extension of* PA *is complete. There must be an arithmetic statement it neither proves nor disproves.*
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We need the restriction. True arithmetic TA—the set of all truths of $\mathbb{N}$—is a complete extension of PA.

(Moreover, the low basis theorem implies that there are complete extensions of PA which are arithmetically definable, specifically, Δ_2 in the arithmetical hierarchy.)

Arithmetization

- Gödel's beta lemma states that arbitrary finite sequences can be coded as a single number, and this is provable within PA.
- Thus any finite mathematical object can be coded in arithmetic.
  - What is a finite semiring? It's a tuple $\langle R, +, \times \rangle$ satisfying certain axioms. Represent R by a sequence of its elements and $+$ and $\times$ by sequences giving their multiplication tables. So you can write an arithmetic formula which expresses "n codes a finite semiring".

$+$	1	a	b	0
1	1	1	1	1
a	1	a	1	a
b	1	1	b	b
0	1	a	b	0

$\times$	1	a	b	0
1	1	a	b	0
a	a	a	0	0
b	b	0	b	0
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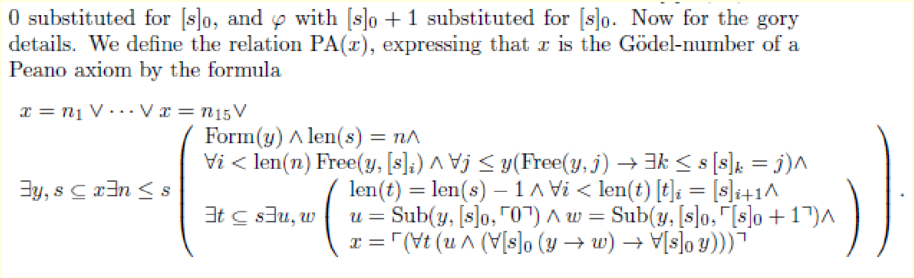
  - More relevant to this talk, objects like Turing machines or logical formulae can be coded in arithmetic.
- Statements like "PA does not prove $0 = 1$" or "such and such Turing machine halts" can be cast as statements in arithmetic.

Arithmetization

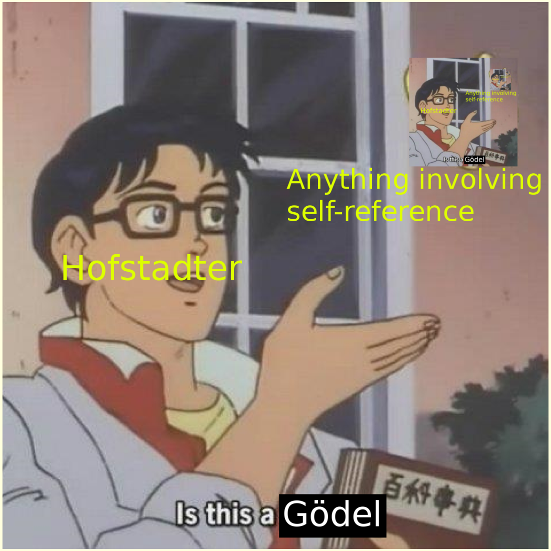


0 substituted for $[s]_0$, and φ with $[s]_0+1$ substituted for $[s]_0$. Now for the gory details. We define the relation $\text{PA}(x)$, expressing that x is the Gödel-number of a Peano axiom by the formula

$$x = n_1 \vee \cdots \vee x = n_{15} \vee$$
$$\exists y, s \subseteq x \exists n \leq s \left(\begin{array}{l} \text{Form}(y) \wedge \text{len}(s) = n \wedge \\ \forall i < \text{len}(n)\, \text{Free}(y,[s]_i) \wedge \forall j \leq y(\text{Free}(y,j) \to \exists k \leq s\, [s]_k = j)\wedge \\ \exists t \subseteq s \exists u, w \left(\begin{array}{l} \text{len}(t) = \text{len}(s) - 1 \wedge \forall i < \text{len}(t)\, [t]_i = [s]_{i+1} \wedge \\ u = \text{Sub}(y,[s]_0, \ulcorner 0 \urcorner) \wedge w = \text{Sub}(y,[s]_0,\ulcorner [s]_0+1\urcorner)\wedge \\ x = \ulcorner(\forall t\,(u \wedge (\forall [s]_0\,(y\to w) \to \forall [s]_0\, y)))\urcorner \end{array}\right)\end{array}\right).$$

(Taken with permission from Victoria Gitman's lecture notes for Mathematical Logic, Spring 2013.)

Self-reference



Self-reference

Anything involving self-reference
Hofstadter
Is this not a Gödel

Self-reference

- The Gödel fixed-point lemma states that a form of self-reference is possible for logical formulae.

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- You can now prove a form of the first incompleteness theorem by considering σ PA-provably equivalent to "σ is not PA-provable".

- Suppose PA proves σ. Then PA proves that σ is not provable. Whence PA does not prove σ. $\lightning$

  Suppose PA proves $\neg\sigma$. Then PA proves that σ is provable. Whence PA does prove σ. $\lightning$

  (You need an additional lemma for those whences, one we will see on Slide 19. If you've heard of "ω-consistency", this is where it shows up.)

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Some people say the incompleteness theorems are difficult to prove. But if you handwave over the actual hard parts you can fit the proof on one slide. :)

Self-reference

- Also have self-reference for computability theory, via the Kleene recursion theorem. Informally, Turing machines can refer to themselves.

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A fun application: programming languages admit quines—programs that output their own source code.

```
;; Quine in Common Lisp
((lambda (x) (list x (list 'quote x)))
'(lambda (x) (list x (list 'quote x))))
```

Incompleteness and Turing machines

The incompleteness theorems can be recast as saying that whether certain Turing machines halt is undecidable.

A TM p:

- Look at all length 1 proofs from the first 1 axiom of PA.
- Then look at all length 2 proofs from the first 2 axioms of PA.
- $\vdots$
- If at any point you see a proof that ends with $0 = 1$, halt and output affirmatively.

Whether p halts is independent of PA.

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- Adam Yedidia and Scott Aaronson do even better.
- They constructed a TM of size 7910 so that whether it halts is independent of ZFC, but ZFC + large cardinals does prove it halts.

(Specifically an ineffable cardinal will do.)

If you liked Gödel's incompleteness theorems, you'll love his completeness theorem

Theorem (Gödel's Completeness Theorem)

1. *A set of axioms T is consistent if and only if there is a structure satisfying T.*
2. *φ is true in every structure satisfying T if and only if φ is a theorem of T.*

(This is for axioms in first-order logic.)

- This lets us translate talk about proofs, consistency, etc. to talk about structures.
- The incompleteness theorems plus the completeness theorem together imply there must be non-isomorphic structures satisfying the axioms of arithmetic.

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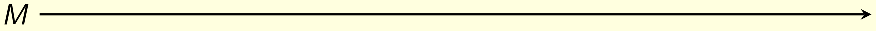
What could these even look like???

Nonstandard models of arithmetic

A model of (Peano) arithmetic is a discretely ordered semiring whose definable subsets are inductive.



- $X \subseteq M$ is definable if you can express $x \in X$ just by quantifying over the elements of M and using the semiring operations and order of M.
- $X \subseteq M$ is inductive if $0 \in X$ and $a \in X \Rightarrow a + 1 \in X$ implies $X = M$.

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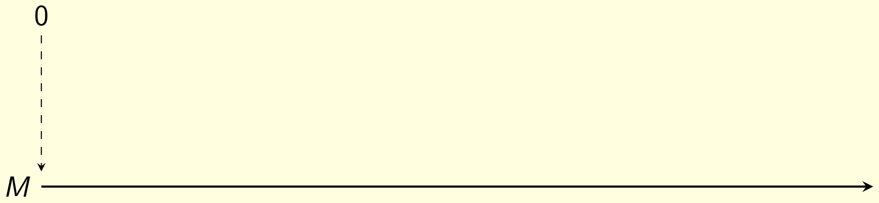

M has a least element 0^M (= the additive identity for M) because the set $\{x \in M : x \geq 0^M\}$ satisfies the inductive hypotheses.

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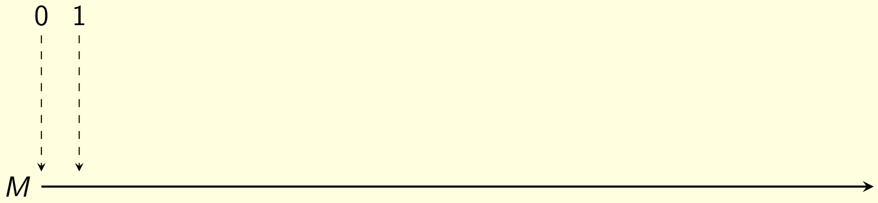

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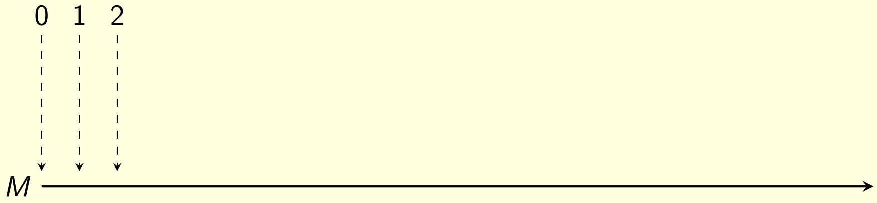

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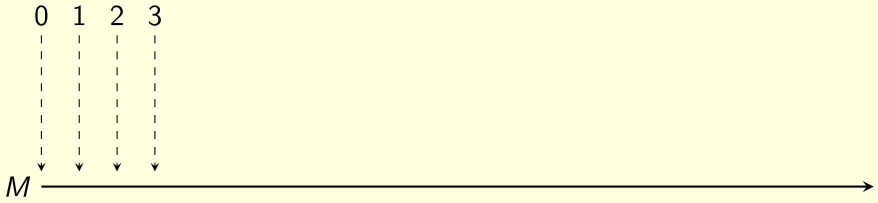

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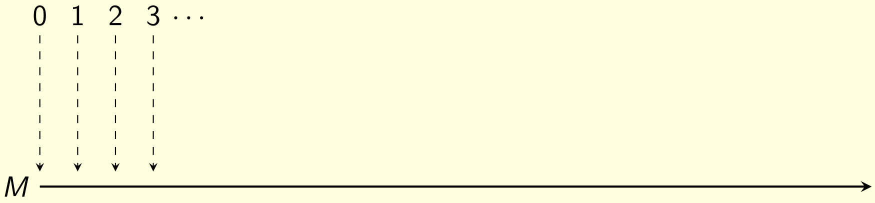

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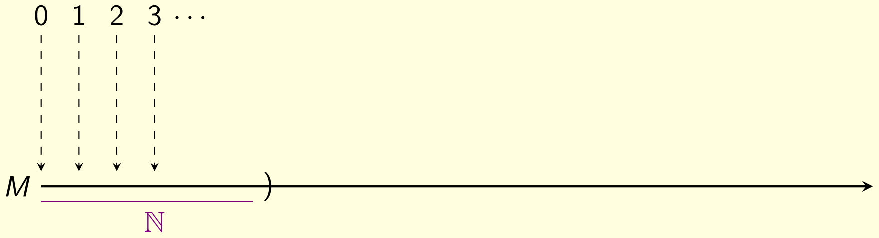

$\mathbb{N}$ embeds as an initial segment on any model of arithmetic.

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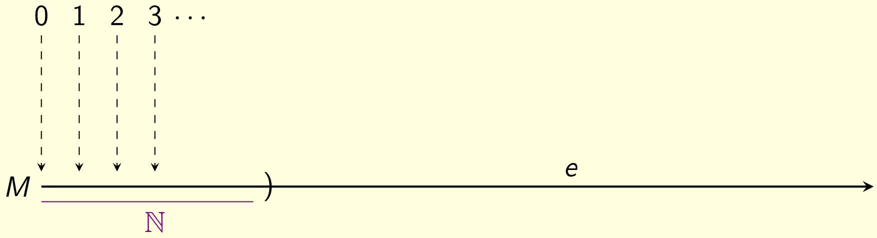

If $e \in M \setminus \mathbb{N}$ then $e > n$ for all $n \in \mathbb{N}$.

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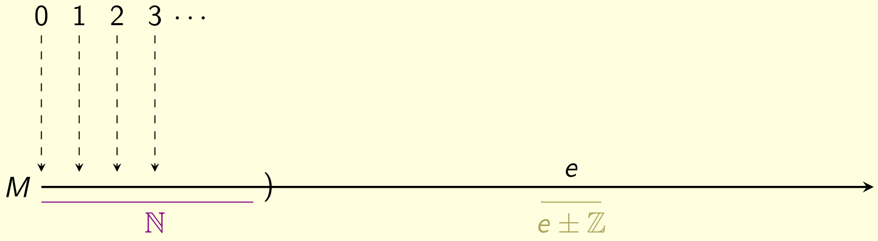

All non-zero elements have a predecessor because

$$\{0\} \cup \{a \in M : a \text{ has a predecessor}\}$$

satisfies the induction hypotheses.

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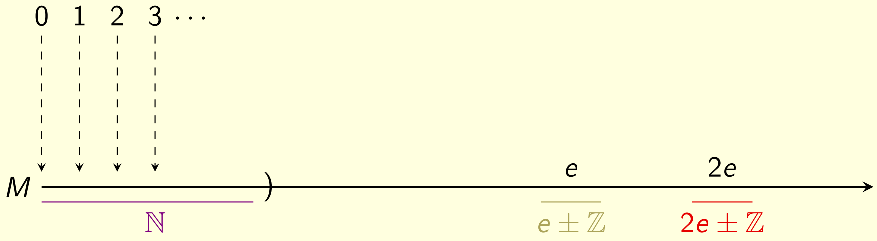

$e + n < e + e = 2e$ for all $n \in \mathbb{N}$.

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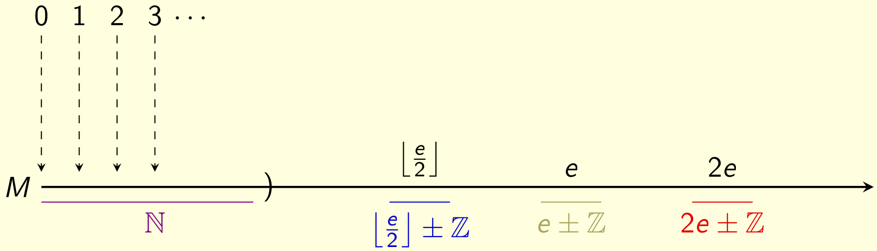

$0 \quad 1 \quad 2 \quad 3 \cdots$
M
$\mathbb{N}$
$\lfloor \frac{e}{2} \rfloor$
$\lfloor \frac{e}{2} \rfloor \pm \mathbb{Z}$
e
$e \pm \mathbb{Z}$
$2e$
$2e \pm \mathbb{Z}$

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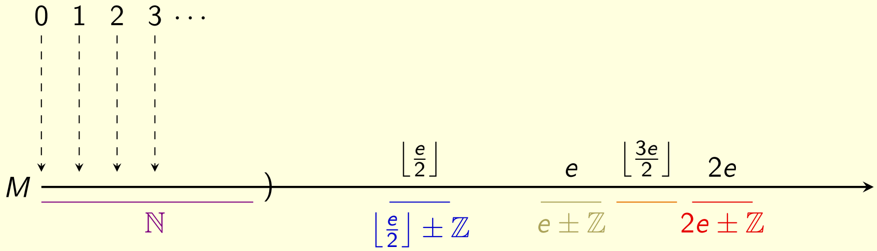

$0 \quad 1 \quad 2 \quad 3 \cdots$
M
$\mathbb{N}$
$\left\lfloor \frac{e}{2} \right\rfloor$
$\left\lfloor \frac{e}{2} \right\rfloor \pm \mathbb{Z}$
e
$e \pm \mathbb{Z}$
$\left\lfloor \frac{3e}{2} \right\rfloor$
$2e$
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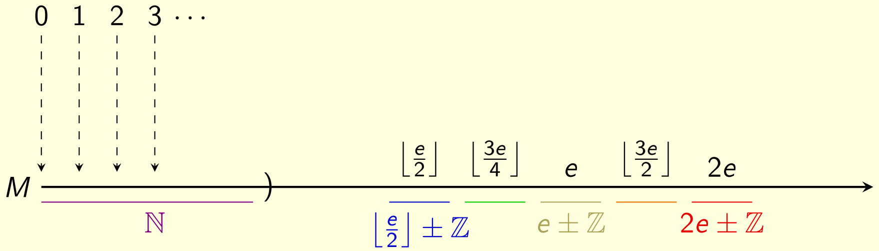
$0 \quad 1 \quad 2 \quad 3 \cdots$
M
$\mathbb{N}$
$\left\lfloor \frac{e}{2} \right\rfloor$
$\left\lfloor \frac{3e}{4} \right\rfloor$
e
$\left\lfloor \frac{3e}{2} \right\rfloor$
$2e$
$\left\lfloor \frac{e}{2} \right\rfloor \pm \mathbb{Z}$
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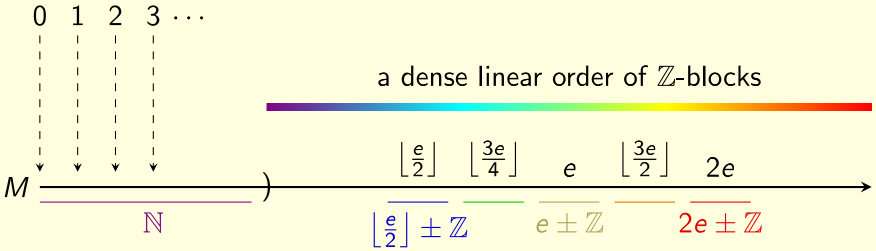

$0 \quad 1 \quad 2 \quad 3 \cdots$
a dense linear order of $\mathbb{Z}$-blocks
$\left\lfloor \frac{e}{2} \right\rfloor$
$\left\lfloor \frac{3e}{4} \right\rfloor$
e
$\left\lfloor \frac{3e}{2} \right\rfloor$
$2e$
M
$\mathbb{N}$
$\left\lfloor \frac{e}{2} \right\rfloor \pm \mathbb{Z}$
$e \pm \mathbb{Z}$
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Facts about nonstandard models of arithmetic

- First constructed by Thoralf Skolem. (Skolem used an ultrapower construction.)
- There are many different nonisomorphic models of arithmetic of any infinite cardinality. In particular, there are $2^{\aleph_0}$ isomorphism classes for countable models of arithmetic.

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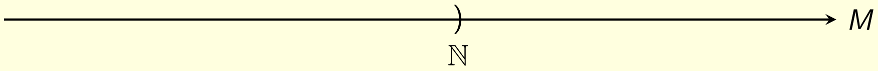
- Open Question (Harvey Friedman): $\mathbb{N}$ has the property that if a model of arithmetic is order-isomorphic to it then they are fully isomorphic. Does any other model of arithmetic have this property?
- (Stanley Tennenbaum) If M is nonstandard then neither the $+$ nor $\times$ of M is a computable function.

Turing machines in a nonstandard world



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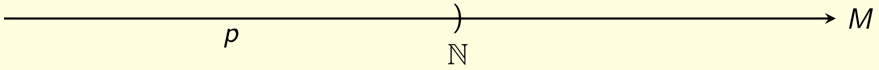

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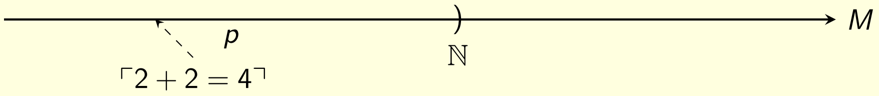

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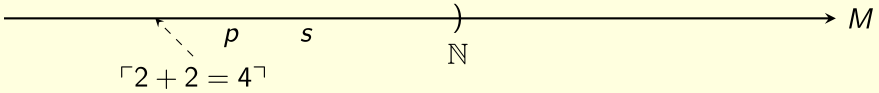

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- s is a computation log witnessing that p outputs $\ulcorner 2+2=4 \urcorner$.

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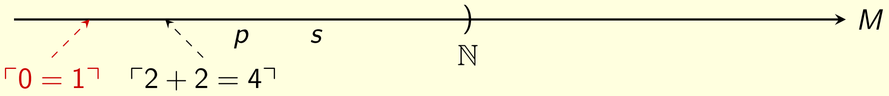

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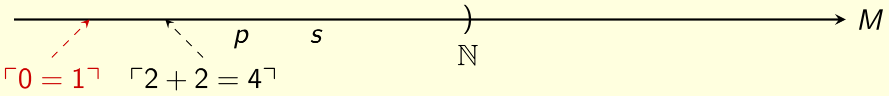
  (s is a number coding the sequence of computation steps. Checking that s has this property only requires looking in $\mathbb{N}$.)
- If we run p in $\mathbb{N}$, then we never output $\ulcorner 0 = 1 \urcorner$.

Turing machines in a nonstandard world



- Consider p the TM which enumerates the theorems of arithmetic.
- s is a computation log witnessing that p outputs $\ulcorner 2 + 2 = 4 \urcorner$.

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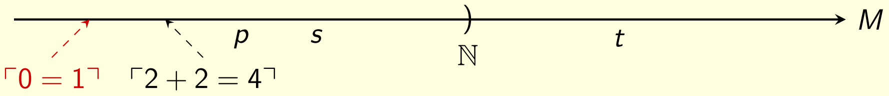

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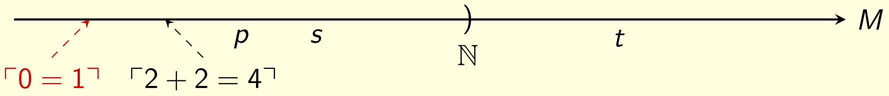

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- Then there is a computation log t witnessing that p outputs $\ulcorner 0 = 1 \urcorner$. But t must be nonstandard! In other words, we had to run p for a nonstandard number of steps to output $\ulcorner 0 = 1 \urcorner$.
- The point: By moving to a larger world we made p output more numbers.

The absoluteness of computability

In summary:

- The statement "the TM p outputs n for some input" is upward absolute—if it's true it stays true if we end-extend to a larger model.

  (Logicians call this sort of statement a Σ_1 statement. By the MRDP theorem, these are the statements equivalent to one whose only quantifiers are a block of $\exists$s.)

- But the statement "the TM p does not output n for some input" is not upward absolute. (It is downward absolute though.)

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  By Gödel's completeness theorem plus the last slide, Peano arithmetic proves every true (i.e. in $\mathbb{N}$) statement of this form.
- But the statement "the TM p does not output n for some input" is not upward absolute. (It is downward absolute though.)

  (Logicians call this sort of statement a Π_1 statement.)

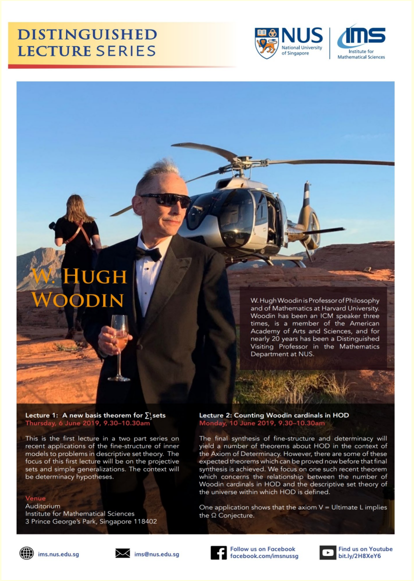
  Both the first and second incompleteness theorems are about statements of this form.

Woodin's universal algorithm

We've seen that the behavior of a Turing machine can be undecidable.

- Proof theoretic: It may be independent of PA how p behaves.
- Model theoretic: Running p in different nonstandard models of arithmetic may produce different behavior.

I want to talk about a striking case of the undecidability of how Turing machines behave, due to W. Hugh Woodin, where p can output anything at all if run in the right universe!


DISTINGUISHED LECTURE SERIES

NUS National University of Singapore | IMS Institute for Mathematical Sciences

W. HUGH WOODIN

W. Hugh Woodin is Professor of Philosophy and of Mathematics at Harvard University. Woodin has been an ICM speaker three times, is a member of the American Academy of Arts and Sciences, and for nearly 20 years has been a Distinguished Visiting Professor in the Mathematics Department at NUS.

Lecture 1: A new basis theorem for Σ^1_3 sets
Thursday, 6 June 2019, 9.30–10.30am

This is the first lecture in a two part series on recent applications of the fine-structure of inner models to problems in descriptive set theory. The focus of this first lecture will be on the projective sets and simple generalizations. The context will be determinacy hypotheses.

Venue
Auditorium
Institute for Mathematical Sciences
3 Prince George's Park, Singapore 118402

Lecture 2: Counting Woodin cardinals in HOD
Monday, 10 June 2019, 9.30–10.30am

The final synthesis of fine-structure and determinacy will yield a number of theorems about HOD in the context of the Axiom of Determinacy. However, there are some of these expected theorems which can be proved now before that final synthesis is achieved. We focus on one such recent theorem which concerns the relationship between the number of Woodin cardinals in HOD and the descriptive set theory of the universe within which HOD is defined.

One application shows that the axiom V = Ultimate L implies the Ω Conjecture.

ims.nus.edu.sg
ims@nus.edu.sg
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Woodin's universal algorithm, first form

Theorem (Woodin)

There is a Turing machine p with the following properties.

1. *p provably enumerates a finite sequence.*
2. *Running p inside $\mathbb{N}$ never produces any output, i.e. it enumerates the empty sequence.*
3. *But, for any finite sequence s of natural numbers there is a nonstandard model of arithmetic M so that running p in M enumerates exactly s.*

Woodin's algorithm

(This construction for Woodin's theorem is due to Joel David Hamkins.)

The Turing machine p:

- p searches through the proofs of Peano arithmetic, looking at the theorems they prove.
- p is looking for a theorem of the form "p does **not** enumerate the sequence s", for s some nonempty finite sequence of numbers.

  (p can refer to itself by the Kleene Recursion theorem.)
- If p ever sees this, then p outputs the sequence s.

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Claim: Run in $\mathbb{N}$, p outputs the empty sequence.

Otherwise p outputs some s. So Peano arithmetic proves this true Σ_1 statement. But by the definition of p, this also means that Peano arithmetic proves that p does not output s. This would mean that Peano arithmetic is inconsistent. But it's not.

Checking the extension property

Definition (The Turing machine p)

- p searches through the proofs of Peano arithmetic, looking for a theorem of the form "p does **not** enumerate the sequence s", for s some nonempty sequence of numbers.
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Fix a finite sequence of natural numbers s. We want to find a nonstandard model of arithmetic M in which running p outputs s.

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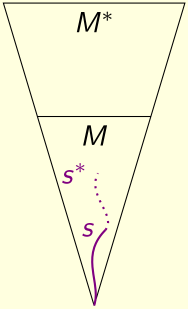
So by Gödel's completeness theorem we can find a model of arithmetic in which p outputs s. $\square$

Woodin's universal algorithm, general form

Theorem (Woodin)

There is a Turing machine p with the following properties.

1. *p provably enumerates a finite sequence.*
2. *Running p inside $\mathbb{N}$ never produces any output, i.e. it enumerates the empty sequence.*
3. *Suppose M a model of arithmetic in which p enumerates s and that s^* is a sequence in M which extends s. Then we can end-extend M to a larger model of arithmetic M^* in which p enumerates s^*.*



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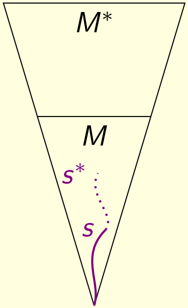

Proof idea: Do a similar argument, but internally to M. Need some more technical lemmata to check that the argument can be arithmetized.

Aside: What about the fourth pillar of mathematical logic?

Traditionally mathematical logic has been divided into four pillars: computability theory, proof theory, model theory, and set theory.*

So far in this talk we've seen the first three of these. What about the fourth?

* (Recent developments e.g. in categorical logic undermine this taxonomy.)

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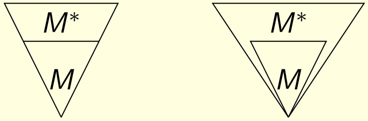
Analogous to Woodin's universal algorithm in arithmetic there are results in set theory, where there's more than one sensible notion of extension to consider. In set theory: Hamkins and Woodin construct a universal sequence for rank-extensions, and Hamkins and Williams construct a universal sequence for end-extensions.



rank-extension end-extension

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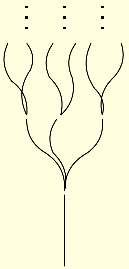
Arithmetic potentialism



- Imagine climbing through the tree of nonstandard models of arithmetic, continually end-extending.
- This potentialist system gives a nonstandard twist on Aristotle's notion of the potential infinite.
- There is a natural interpretation in modal logic—extend ordinary logic by adding two new operators
  - $\Box\varphi$ means φ is necessarily true—true in all extensions.
  - $\Diamond\varphi$ means φ is possibly true—true in some extension.
- (Hamkins) Can use Woodin's universal algorithm to calculate which modal assertions are valid (true in any world under any substitution of variables).

  (Namely, those in the modal theory S4.)
- (Hamkins) There are models of arithmetic which satisfy the maximality principle—if $\Diamond\Box\varphi$ then φ.

Thank you for letting me talk math with my friends!