The geology of inner mantles

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(Partly joint work with Jonas Reitz)

0. Introduction
   A crash course in geology
1. Positive Results
2. Negative Results

Question (Reitz)**:** What if forcing, but backward?

Geology 101

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Definition

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Theorem (Laver, Woodin)

(Over ZFC) The grounds are uniformly first-order definable.

Theorem (Usuba)

(Over ZFC) The grounds are strongly downward directed. If you take a set-sized collection $\{W_i : i \in I\}$ of grounds then there is a ground $W \subseteq W_i$ for all $i \in I$.

Aside: geology in a choiceless universe

Open question: Are the grounds uniformly first-order definable over ZF?

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- (Gitman–Johnstone) If V is an extension of W by a poset of cardinality $\leq \delta$ and $W \models \mathsf{DC}_\delta$ then W is definable in V.
- (Usuba) If there is a proper class of Löwenheim–Skolem cardinals then the grounds are uniformly first-order definable.
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The mantle

The mantle $\mathbb{M}$ is the intersection of the grounds. It is an inner model of ZFC. Indeed, it is the largest (set) forcing invariant inner model.

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Examples:

- $\mathrm{M}^{\mathrm{L}} = \mathrm{L}$.
- If you do nontrivial set forcing over L then $\mathrm{M}^{\mathrm{L}[g]} = \mathrm{L}$ is a ground of $\mathrm{L}[g]$.
- If you force with a class product of Cohen forcings over L then $\mathrm{M}^{\mathrm{L}[G]} = \mathrm{L}$ is not a ground of $\mathrm{L}[G]$.
- (Usuba) If there is an extendable cardinal then M is a ground of V.

How malleable is the mantle?

Simple examples show that M can be V or can be far from V. Can we make a general statement?

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Theorem (Fuchs–Hamkins–Reitz)

There is a class forcing notion which forces the ground model to be the mantle of the extension, and there is a class forcing notion which forces the extension to be its own mantle.

Corollary

The theory of the mantle is ZFC*.*

Warmup: forcing $\mathrm{V} = \mathrm{M}$

Lemma

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To force $\mathrm{V} = \mathrm{M}$ it suffices to code every set of ordinals cofinally often into the continuum pattern.

A slick way to do that: force with an Ord-length iteration of lottery sums to generically pick at each stage α to either make GCH hold at α or fail at α.

- Pick an appropriately spaced out and absolutely definable coding region R.
- Use a set-support iteration.
- At stage $\alpha \in R$ use $\mathrm{Add}(\alpha, (2^{<\alpha})^{++}) \oplus \mathrm{Add}(\alpha^+, 1)$.

A density argument then shows that $\mathrm{V} = \mathrm{M}$ in the extension.

Forcing $\mathrm{V} = \mathrm{M}^{\mathrm{V}[G]}$

We want to code all of V into the mantle of $\mathrm{V}[G]$, but nothing more.

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Solution: force with a product instead of an iteration!

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Claim

If $x \in \mathrm{V}$ is a set of ordinals then $x \in \mathrm{M}^{\mathrm{V}[G]}$.

By a density argument x is coded cofinally often into the continuum pattern in R.

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Claim

If $x \notin \mathrm{V}$ then $x \notin \mathrm{M}^{\mathrm{V}[G]}$.

Split $\mathbb{P}$ into $\mathbb{P}_{\text{head}} \times \mathbb{P}^{\text{tail}}$, a product of set forcing $\mathbb{P}_{\text{head}}$ and $|x|^+$-closed $\mathbb{P}^{\text{tail}}$. Then x had to be added by $\mathbb{P}_{\text{head}}$. But then $\mathrm{V}[G^{\text{tail}}]$ is a ground which misses x.

Inner Mantles

Every universe is the mantle of some larger universe. In particular, the larger universe of which your universe is the mantle is itself the mantle some even larger universe, and so on. Looking from the downward direction: It is sensible to ask about the mantle of the mantle, the mantle of the mantle of the mantle, and so on. And it is sensible to ask whether these are different.

Definition

The sequence of inner mantles M^η for $\eta \in \mathrm{Ord}$ is defined recursively:

- $\mathrm{M}^0 = \mathrm{V}$
- $\mathrm{M}^{\eta+1} = \mathrm{M}^{\mathrm{M}^\eta}$
- $\mathrm{M}^\gamma = \bigcap_{\eta<\gamma} \mathrm{M}^\eta$ for limit γ

Questions about inner mantles

Question (Fuchs–Hamkins–Reitz)

Is there a class forcing to force the ground model to be the η-th inner mantle of the extension?

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Question (Fuchs–Hamkins–Reitz)

What happens, consistently, at limit stages γ? Specifically:

- *Is it consistent that M^γ is a definable class but does not satisfy AC?*
- *Is it consistent that M^γ is not a definable class but each M^η is for $\eta < \gamma$?*

Aside: iterated HOD

Definition

- $\mathrm{HOD}^0 = \mathrm{V}$
- $\mathrm{HOD}^{\eta+1} = \mathrm{HOD}^{\mathrm{HOD}^\eta}$
- $\mathrm{HOD}^\gamma = \bigcap_{\eta<\gamma} \mathrm{HOD}^\eta$ for limit γ

Theorem (McAloon, Jech, Zadrożny)

Every model is the HOD^η of a class forcing extension.

Theorem (McAloon)

Consistently HOD^ω is a definable inner model of $\neg$AC.

Theorem (Harrington)

Consistently HOD^ω is not a definable class.

0. Introduction
1. Positive Results (Joint with Reitz)
   Creationism for set theoretic geology
2. Negative Results

Every model is the η-th inner mantle of another universe

Theorem (Reitz–W.)

There is a class forcing notion $\mathbb{M}(\eta)$, uniformly definable in a parameter $\eta \in \mathrm{Ord}$, so that forcing with $\mathbb{M}(\eta)$ produces a model $\mathrm{V}[G]$ satisfying

$$\mathrm{V} = (\mathrm{M}^{\eta})^{\mathrm{V}[G]}$$

where $\mathrm{M}^i \supsetneq \mathrm{M}^{i+1}$ for all $i < \eta$.

Overview of the proof

If η is finite, this is easy. Just repeatedly force with the Fuchs–Hamkins–Reitz partial order. Then you get $\mathrm{V}[\vec{G}] = \mathrm{V}[G_1]\cdots[G_\eta]$ satisfying

- $(\mathrm{M}^1)^{\mathrm{V}[\vec{G}]} = \mathrm{V}[G_1]\cdots[G_{\eta-1}]$;
- $(\mathrm{M}^2)^{\mathrm{V}[\vec{G}]} = \mathrm{V}[G_1]\cdots[G_{\eta-2}]$;
- $\vdots$
- $(\mathrm{M}^{\eta-1})^{\mathrm{V}[\vec{G}]} = \mathrm{V}[G_1]$;
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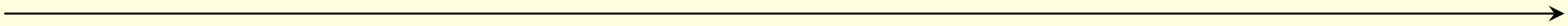
The problem: the order of the inner mantles reverses the order of the iteration. For infinite η, we want to force with an $\eta^\star$-iteration of class products, not an η-iteration.

Set theorists do not have a general theory of iterations on ill-founded orders. But we can handle this specific case.

Defining the forcing $\mathbb{M}(\eta)$

R 

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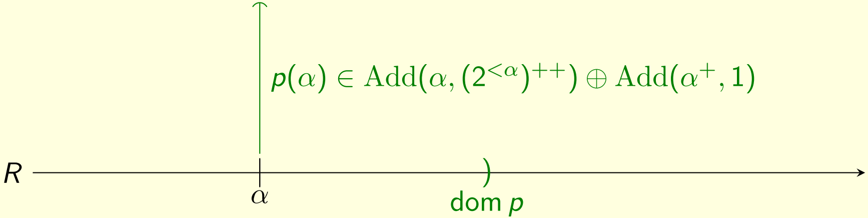

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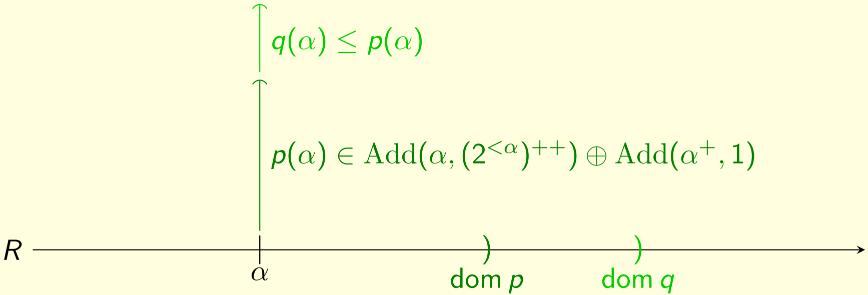

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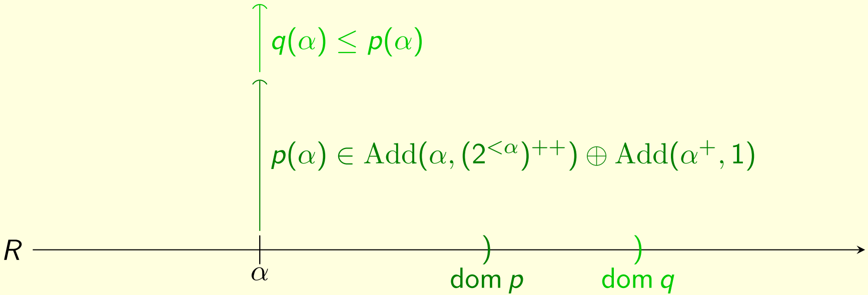

- R is the coding region.
- $q \leq p$.
- Split R into congruence classes R_i for $i < \eta$.
- Then $\langle R_{>i} : i \in \eta \rangle$ is a $\subsetneq$-descending sequence of ordertype η.
- For $\alpha \in R$ let $i(\alpha)$ be the unique i with $\alpha \in R_i$.

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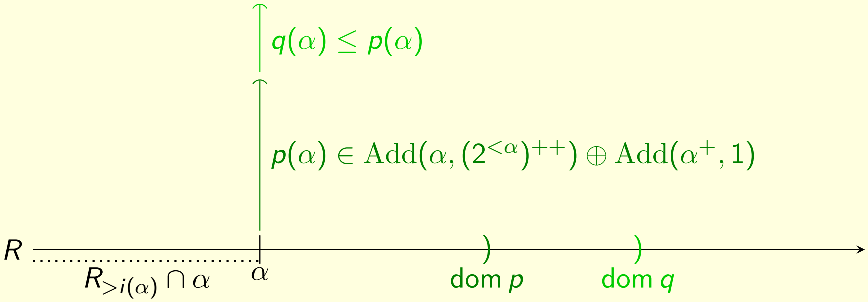

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- Split R into congruence classes R_i for $i < \eta$.
- Then $\langle R_{>i} : i \in \eta \rangle$ is a $\subsetneq$-descending sequence of ordertype η.
- For $\alpha \in R$ let $i(\alpha)$ be the unique i with $\alpha \in R_i$.
- $p(\alpha)$ is a $\mathbb{M}(\eta) \restriction (R_{>i(\alpha)} \cap \alpha)$-name for an appropriate condition.
- $p \restriction (R_{>i(\alpha)} \cap \alpha)$ forces over $\mathbb{M}(\eta) \restriction (R_{>i(\alpha)} \cap \alpha)$ that $q(\alpha) \leq p(\alpha)$.

Defining the forcing $\mathbb{M}(\eta)$

Fix a suitable coding region R. Split R into η many congruence classes R_i. For $\alpha \in R$ let $i(\alpha)$ be the unique $i < \eta$ so that $\alpha \in R_i$. Let $R_{>i}$ have the obvious meaning.

$\mathbb{M}(\eta)$ is the class forcing

- whose conditions are set-sized functions p with domain an initial segment of R
- so that for all $\alpha \in \operatorname{dom} p$ we have $p(\alpha)$ is an $\mathbb{M}(\eta) \upharpoonright (R_{>i(\alpha)} \cap \alpha)$-name for a condition in $\mathrm{Add}(\alpha, (2^{<\alpha})^{++}) \oplus \mathrm{Add}(\alpha^+, 1)$.
- For $p, q \in \mathbb{M}(\eta)$, say that $q \leq p$ if
  - $\operatorname{dom} q \supseteq \operatorname{dom} p$ and
  - for all $\alpha \in \operatorname{dom} p$ we have $p \upharpoonright (R_{>i(\alpha)} \cap \alpha)$ forces over $\mathbb{M}(\eta) \upharpoonright (R_{>i(\alpha)} \cap \alpha)$ that $q(\alpha) \leq p(\alpha)$.

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For later purposes we will need $\mathbb{M}(\eta)$ to be η^+-closed. This is easily arranged by having R only contain cardinals $\geq \eta^+$.

Questions about $\mathbb{M}(\eta)$

- $\mathbb{M}(\eta)$ was defined as a weird iteration of ordertype Ord. In what sense can we think of it as an iteration of ordertype $\eta^\star$?
- What closure/distributivity conditions are satisfied by the stages of $\mathbb{M}(\eta)$?
- Does it even preserve ZFC?

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Using the technology of generalized Cohen iterations we can answer these questions.

- $\mathbb{M}(\eta)$ is a progressively distributive iteration: for $\alpha \in R$, $\mathbb{M}(\eta)$ factors as $\mathbb{M}(\eta) \cong \mathbb{M}(\eta)_{\text{head}} * \mathbb{M}^{\text{tail}}$ where $\mathbb{M}(\eta)_{\text{head}} \Vdash \mathbb{M}^{\text{tail}}$ is α-distributive.
- In particular, $\mathbb{M}(\eta)$ preserves ZFC.
- $\mathbb{M}(\eta)$ preserves R and each R_i.
- The same holds for $\mathbb{M}(\eta) \restriction R_{\geq i}$.

$\mathbb{M}(\eta)$ as an $\eta^\star$-iteration

For notational convenience: set $\mathbb{P} = \mathbb{M}(\eta)$ and $\mathbb{P}_i = \mathbb{M}(\eta) \upharpoonright R_{\geq i}$.

Observation

$$\mathbb{P} = \mathbb{P}_0 \supseteq \mathbb{P}_1 \supseteq \cdots \supseteq \mathbb{P}_i \supseteq \cdots \supseteq \mathbb{P}_\eta \qquad i \leq \eta$$

is a continuous descending chain of class forcing notions, and for $i < j$ we have $\mathbb{P}_j$ is a complete suborder of $\mathbb{P}_i$.
In particular, $\mathbb{P}$ factors as $\mathbb{P}_i * \dot{\mathbb{Q}}^{\text{tail}}$ for each $i < \eta$.

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In particular, $\mathbb{P}$ factors as $\mathbb{P}_i * \dot{\mathbb{Q}}^{\text{tail}}$ for each $i < \eta$.

Let $G \subseteq \mathbb{P}$ be generic over V, and let G_i be the restriction of G to $\mathbb{P}_i$. In particular G_η is the trivial filter over the trivial forcing $\mathbb{P}_\eta$.

Claim

For $i \leq \eta$, $(\mathrm{M}^i)^{\mathrm{V}[G]} = \mathrm{V}[G_i]$.

Prove this by induction.

The successor step

$\mathbb{P}_i$ factors as $\mathbb{P}_{i+1} * \dot{\mathbb{Q}}_i$ where

$$\mathbb{Q}_i = \prod_{\alpha \in R_i} \mathrm{Add}(\alpha, (2^{<\alpha})^{++}) \oplus \mathrm{Add}(\alpha^+, 1).$$

Now do the Fuchs–Hamkins–Reitz argument.

The limit step

Lemma (Jech)

Let i be a limit ordinal and

$$\mathbb{B}_0 \supseteq \mathbb{B}_1 \supseteq \cdots \supseteq \mathbb{B}_j \supseteq \cdots \supseteq \mathbb{B}_i$$

be a continuous descending sequence of complete sub-boolean algebras, where $\mathbb{B}_0$ is i^+-distributive. If $G_0 \subseteq \mathbb{B}_0$ is generic over V and $X \in V[G_j]$ for all $j < i$, then $X \in V[G_i]$.

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But if $\mathbb{P}$ is a progressively distributive iteration, factoring as $\mathbb{Q}_\alpha * \mathbb{Q}^{\text{tail}}$ for arbitrarily large α so that the $\mathbb{P}_j \cap \mathbb{Q}_\alpha$ form a chain like in Jech's lemma, then we get the conclusion of Jech's lemma.

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This is where we use the assumption that $\mathbb{P} = \mathbb{M}(\eta)$ is η^+-closed!

What's the deal with limit stages?

Observation

If M^{η} *is a definable inner model of* ZFC, *then so is* $\mathrm{M}^{\eta+1}$.

Question

If M^{η} *is a definable inner model of* ZFC *for all* $\eta < \gamma$ *for limit* γ, *must also the same hold for* M^{γ}*?*

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Observation

Over Gödel–Bernays second-order set theory, $\mathsf{ETR}_{<\mathrm{Ord}}$*, the principle of Elementary Transfinite Recursion for recursions of length* $<\mathrm{Ord}$*, implies that* M^{η} *is a class for all ordinals* η.

What's the deal with limit stages?

Proposition

If the sequence $\langle \mathrm{M}^\eta : \eta < \gamma \rangle$ *is definable for limit* γ *and each* $\mathrm{M}^\eta \models$ ZFC*, then* $\mathrm{M}^\gamma \models$ ZF*.*

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There are two possibilities for what could go wrong at limit stages:

- M^γ is definable, but does not satisfy AC.
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Both are possible, at least in case $\gamma = \omega$. The arguments use ideas from the analogous results about HOD^ω.

Different ways to code sets into inner mantles

For the positive results, Reitz and I used continuum coding—sets of ordinals are coded by the pattern of where GCH holds.

For the forthcoming results it will be convenient to use a different coding, call it Cohen coding. For this, sets of ordinals are coded by the pattern of which cardinals have subsets which are Cohen-generic over L.

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Of course, this coding is applicable in fewer universes of sets. We need a coding region which is clean for coding—no Cohen sets on those cardinals. But since the goal is to build counterexample models we can use a more restrictive coding, and it simplifies some arguments.

Warmup: coding a set into $\mathbb{M}$

For the positive results, we let the generic pick where to code, and by a density argument every set of ordinals was coded. Now we want to be a bit more precise.

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Lemma

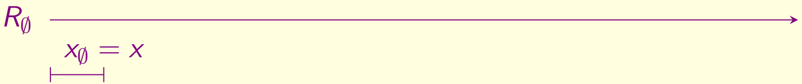
Suppose W is a ground of V. Then W and V agree on a tail about which cardinals contain Cohen generics (over L).

Suppose we are in a model appropriate for Cohen coding, and x is a set of ordinals with $\lambda = \sup x \notin x$. (This is a convenience we assume without loss of generality.) Let $R \subseteq \mathrm{Ord}$ be an absolutely definable class which is clean for Cohen coding.

Force with the product of $\mathrm{Add}(\alpha, 1)$ for each $\alpha \in R$ which is the $(\lambda \cdot \xi + i)$-th cardinal in R for some $\xi \in \mathrm{Ord}$ and some $i \in x$.

In the forcing extension, x is coded cofinally often in R at the blocks $[\lambda \cdot \xi, \lambda \cdot \xi + \lambda)$, and so $x \in \mathrm{M}$.

Tree-like coding to get a set into M^k



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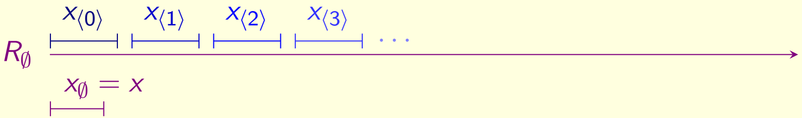

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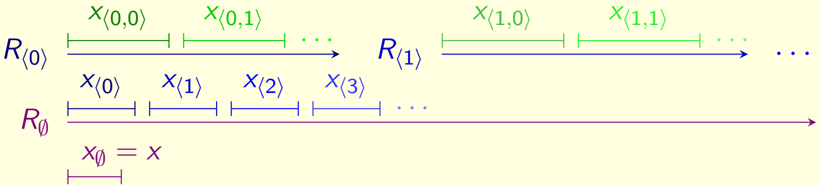
Tree-like coding to get a set into M^k



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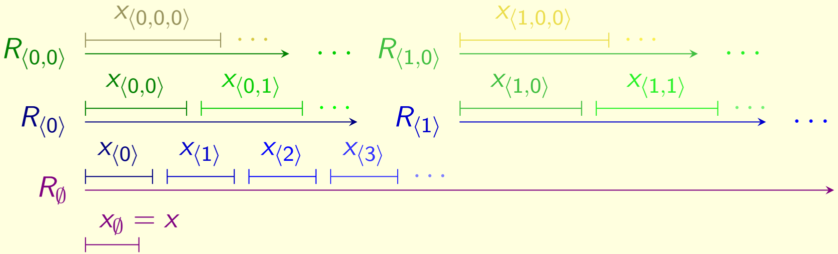

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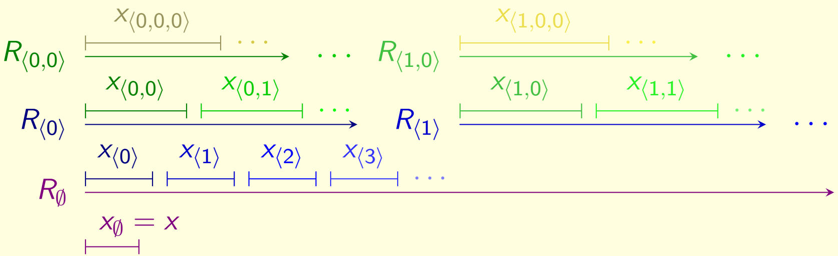

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- Continue upward to code all x_s in R_s for $s \in {}^{<k}\mathrm{Ord}$.
- This is a k-step iteration of class products, call it k-height tree-like coding or $\mathbb{T}(k, R, x)$.

Tree-like coding to get a set into M^k

- Digging deeper through inner mantles corresponds to climbing down the tree. After forcing with $\mathbb{T}(k, R, x)$: For each $\ell \leq k$ and x_s, we get $x_s \in \mathrm{M}^\ell$ if and only if $\operatorname{len} s \leq k - \ell$.
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- $\mathbb{T}(k, R, x)$ and $\mathbb{T}(\ell, S, y)$ don't interfere with each other, if R and S are disjoint.
- $\mathbb{T}(k, R, x)$ is uniformly definable in k, R, x. So if you have a uniform listing of k's, R's, and x's then you can define the product of the $\mathbb{T}(k, R, x)$'s.

Consistently $\mathrm{M}^\omega \not\models$ AC

Theorem (W.)

There is a class forcing extension of L in which M^ω is a definable inner model of ZF $+ \neg$AC. *Specifically, there is no well-order of $\mathcal{P}(\omega)$ in this extension.*

Sketch of the argument (following McAloon's HOD^ω)

- Start with L.
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- Let G_k be the portion of G corresponding to sequences which are at least k many levels from the top of their tree, so

$$G = G_0 \supsetneq G_1 \supsetneq \cdots \supsetneq G_k \supsetneq \cdots$$

- Inductively show that $(\mathrm{M}^k)^{\mathrm{L}[A][G]} = \mathrm{L}[G_k]$.

Sketch of the argument (following McAloon's HOD^ω)

- Observe: each row of A is obtained from A_k by prepending finitely many bits to a row, so each row of A is in M^ω.
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- So M^ω has a well-order of $\mathcal{P}(\omega)$ iff it has one of ordertype ω_1 iff there is $x \subseteq \omega_1$ in M^ω so that $\mathcal{P}(\omega) \cap \mathrm{M}^\omega \in \mathrm{L}[x]$.

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- But $\mathrm{Add}(\omega, \omega_1)$ has the ccc, so $x \in \mathrm{L}[A \upharpoonright \alpha]$ for some countable α, the restriction of A to the first α many rows.
- But then if z is a row of A above α then $z \notin \mathrm{L}[x]$. So z witnesses that $\mathcal{P}(\omega) \cap \mathrm{M}^\omega \notin \mathrm{L}[x]$.

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M^ω may fail to be a definable class

Theorem (W.)

There is a class forcing extension of L in which the satisfaction predicate for L is definable over its M^ω.

Corollary

There are models of ZFC whose M^ω is not a definable class.

Paris models and coding truth

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- (Paris) Every consistent completion of ZF has a Paris model.
- Suppose $L \models \mathrm{ZFC} + \mathrm{V} = \mathrm{L}$ is a Paris model.
  - Then any outer model $V \models \mathrm{ZFC}$ of L is a Paris model.
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So if V is an extension of a Paris model of $\mathrm{V} = \mathrm{L}$ whose ω-th mantle can define the satisfaction predicate for L then the ω-th mantle is not a definable class in V.

More about coding sets into inner mantles

Idea to prove the theorem (Harrington): Assign to each formula φ with parameters from L a cardinal $\kappa(\varphi)$. Then code so that M^ω has a Cohen subset of $\kappa(\varphi)$ iff $\mathrm{L} \models \varphi$. We need to define the coding forcing using only a bounded level of truth in L to ensure that the forcing is definable.

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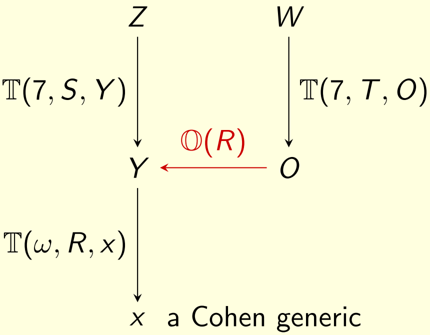
For this we will need more coding tools.

- If you force with $\mathbb{T}(\omega, R, x)$, ω-height tree-like coding, then every piece x_s of the generic will be in M^1, M^2, and so on.
- You can do $\mathbb{T}(k, R, X)$ or $\mathbb{T}(\omega, R, X)$ for a proper class X, say by breaking X into set-sized chunks and coding the chunks on subregions of R.
- You can overwrite a coding block R by adding a Cohen generic to every $\alpha \in R$. Let $\mathbb{O}(R)$ be the overwrite forcing for R.

A toy example of more complicated coding



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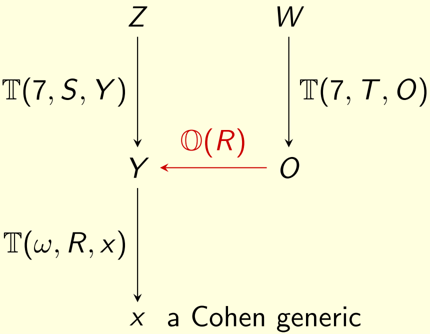

The code W ensures O survives to M^7, overwriting the region R where the code Y lives. Nevertheless, before we dig past M^7 we can recover Y using the code Z. Namely, Z ensures that Y is in M^7, which in turn ensures that $x \in \mathrm{M}^8$. But in M^7 we no longer have a code for Y, and the coding region was overwritten. So in M^8 we have that x is no longer Cohen coded, and thus $x \notin \mathrm{M}^9$.

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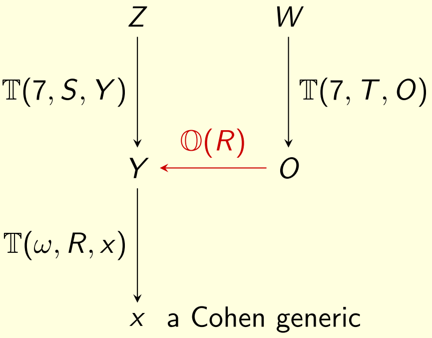

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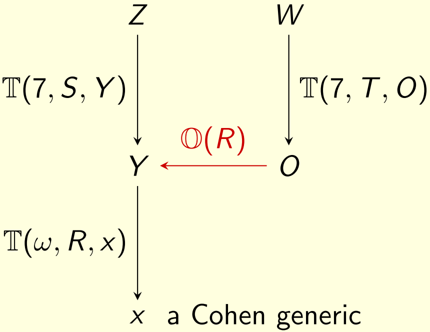

Why not just use $\mathbb{T}(8, R, x)$?

The point: O is what kept x out of M^9. If we had in turn overwritten the code W for O then we would've ensured $x \in \mathrm{M}^\omega$.

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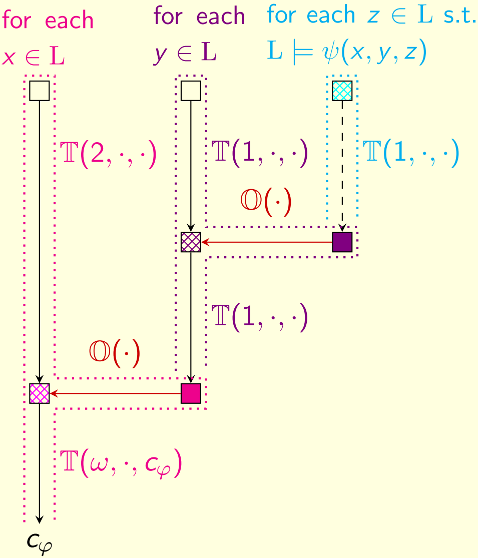
Triangle coding—do for every φ with parameters from L



- $\varphi = \exists x \neg \exists y \exists z \; \psi(x, y, z)$ is Σ_3.
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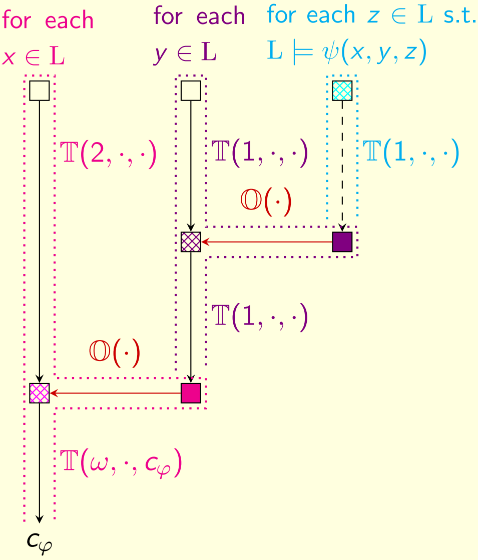

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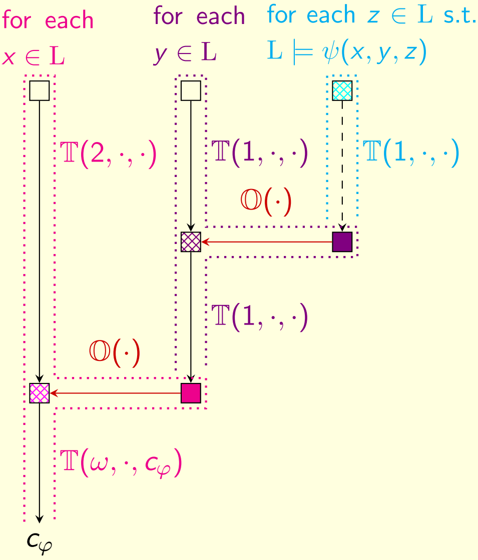

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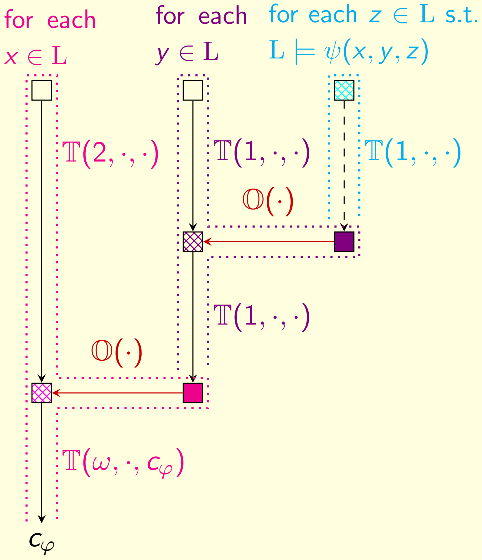

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- ■ survives into M^2 iff for some y ▩ is not overwritten in M^1.

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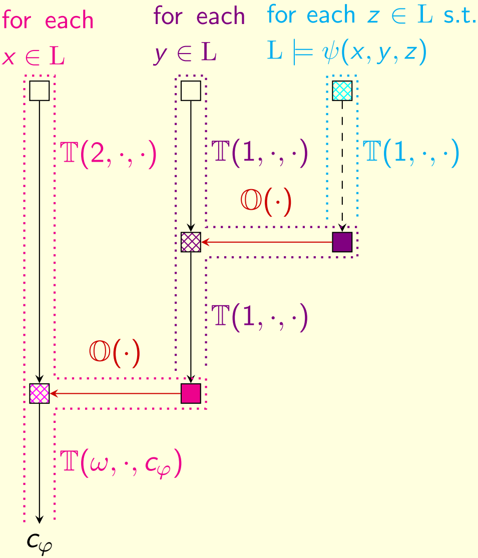

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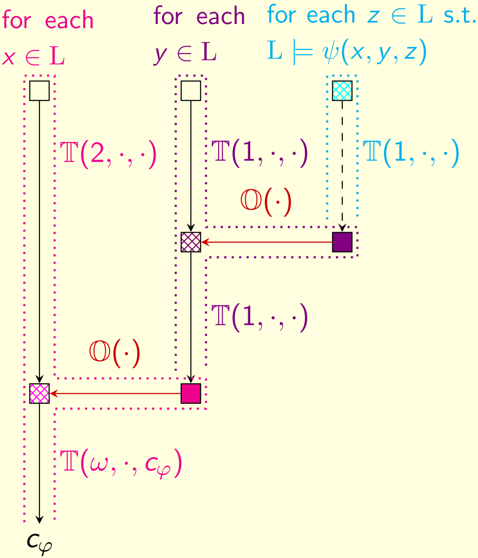

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- Always survives to M^2 and survives to M^1.
- Fix $x \in \mathrm{L}$. We want to use to get c_φ is in M^ω, but in M^2 is overwritten if survives.
- survives into M^2 iff for some y is not overwritten in M^1.
- is overwritten in M^1 iff gets into M^1 iff for some z exists.
- Altogether: c_φ gets into M^ω iff $\exists x \in \mathrm{L}$ so that $\neg\exists y \in \mathrm{L}$ so that $\exists z \in \mathrm{L}$ so that $\mathrm{L} \models \psi(x, y, z)$.

The general case

Conjecture

Let γ be a limit ordinal. Then there is a class forcing extension of L in which M^γ can define the satisfaction predicate for L, and M^η is a definable class for each $\eta < \gamma$.

Thank you!

- Fuchs, Hamkins, and Reitz, “Set theoretic geology”. Annals of Pure and Applied Logic, Vol. 166, No. 4, pp. 464–501 (2015).
- Reitz, “Cohen forcing and inner models”. Mathematical Logic Quarterly, Vol. 66, No. 1, pp. 65–72 (2020).
- Reitz and Williams, “Inner mantles and iterated HOD”. Mathematical Logic Quarterly, Vol. 65, No. 4, pp. 498–510 (2019).
- Zadrożny, “Iterating ordinal definability”. Annals of Pure and Applied Logic, Vol. 24, No. 3, pp. 263–310 (1983).