The Σ_1 universal finite sequence

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7th biannual European Set Theory Conference
2019 July 4

Joint work with Joel David Hamkins and Philip Welch

Potentialism as a general framework

Definition

A potentialist system $(\mathcal{M}, \subseteq)$ is a collection of worlds which are structures in a common signature, ordered by a reflexive, transitive relation $\subseteq$ which refines the substructure relation.

Used to formalize the intuition of a dynamically growing domain.

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For instance, Linnebo and Stewart used this framework to formalize Aristotle's notion of the potential infinite. In their potentialist system, worlds are finite initial segments of ω, ordered by extension.

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- The hyperverse of countable transitive models Worlds are countable transitive models.

A modal interpretation

$(\mathcal{M}, \subseteq)$ is a potentialist system.

- $\Diamond\varphi$ is true at a world M if φ is true in some extension $N \supseteq M$.
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The theory S4 is always a lower bound for the modal validities.

$$\Box(p \Rightarrow q) \Rightarrow (\Box p \Rightarrow \Box q)$$

$$\neg\Diamond p \Leftrightarrow \Box\neg p$$

$$\Box p \Rightarrow p$$

$$\Box p \Rightarrow \Box\Box p$$

The modal logic of potentialism

- Zermeloian potentialism has S4.3 as its modal validities. (Hamkins–Linnebo)
- The generic multiverse has S4.2 as its modal validities. (Hamkins–Löwe)
- The hyperverse of countable transitive models has S4.2 as its modal validities. (Hamkins–Linnebo)

$$(.2) \qquad \Diamond\Box\varphi \Rightarrow \Box\Diamond\varphi$$

$$(.3) \qquad (\Diamond\varphi \wedge \Diamond\psi) \Rightarrow ((\varphi \wedge \Diamond\psi) \vee (\Diamond\varphi \wedge \psi))$$

Branching versus directed potentialism

Having S4.2 as modal validities expresses directedness of the modalities, while failures of the .2 axiom express that there is incompatible branching.

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Warning! Directedness/branching of the modalities is not the same thing as directedness/branching of the order relation on the potentialist system. The generic multiverse is not directed as a partial order (Mostowski), but the modal validities for forcing potentialism are precisely S4.2 (Hamkins–Löwe).

End-extensional potentialism

The potentialist system we consider: countable models of set theory, ordered by end-extension.

Informally: N end-extends M if $M \subseteq N$ and old sets have no new elements.

Formally: N end-extends M if $M \subseteq N$ and $a \in^N b \in M$ implies $a \in M$.

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Observation

If N end-extends M and $M \models \varphi(a)$ for a Σ_1 formula φ, then $N \models \varphi(a)$.

The Σ_1-definable universal finite sequence

Let $\overline{\mathrm{ZF}}$ be a fixed computably enumerable extension of ZF.

Theorem (Hamkins–Welch–W.)

There is a Σ_1 definition for a finite sequence

$$a_0, \ldots, a_n$$

with the following properties.

1. ZF *proves the sequence is finite.*
2. *If $M \models \overline{\mathrm{ZF}}$ is transitive then the sequence in M is the empty sequence.*
3. *If in countable $M \models \overline{\mathrm{ZF}}$ the sequence is s and $t \in M$ is any finite extension of s, then there is $N \models \overline{\mathrm{ZF}}$ an end-extension of M so that the sequence in N is exactly t.*

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4. *Indeed, in (3) it suffices that $M \models \mathrm{ZF}$ has an inner model $W \models \overline{\mathrm{ZF}}$.*

Process A—for ω-nonstandard models

$a_0, \ldots, a_n$ is defined using auxiliary information $k_0 > \cdots > k_n$ finite ordinals and $m_0 \in \cdots \in m_n$ countable transitive sets.

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To extend: if stage n fails in M then for any $a \in M$ and nonstandard $k < k_{n-1}$ can find in $M^+[g]$, a forcing extension of an elementary end-extension of M, a model of $\overline{\mathrm{ZF}}_k$, which end-extends $m = {V_\theta}^{M^+} \supseteq M$ and whose process A sequence is $a_0, \ldots, a_{n-1}, a$.

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To extend: again find the desired end-extension of M in a forcing extension of an elementary end-extension of M.

Process C—for all models

You can combine processes A and B into a single process which has the extension property for any countable model of $\overline{\mathsf{ZF}}$.

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Our theorem gives a new proof of the Barwise extension theorem, which does not go through the theory of admissible sets. (Either derive it as an immediate corollary of (4) using $\overline{\mathsf{ZF}} = \mathsf{ZFC} + \mathrm{V} = \mathrm{L}$, or you can give a direct proof similar to our proof for the universal finite sequence.)

Universal finite sequences in a general framework

$(\mathcal{M}, \subseteq)$ a potentialist system, worlds have sequences and integers.

Definition

$\mathcal{M}$ has a universal finite sequence if there is a definition for a sequence s so that:

1. If M is a world in $\mathcal{M}$ then s^M is a finite sequence.
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- Woodin's universal algorithm gives a universal finite sequence for arithmetic potentialism.

A universal finite sequence implies branching potentialism

Theorem (Hamkins)

If a potentialist system admits a universal finite sequence, then its modal validities are precisely S4. *So it has branching modalities.*
(If worlds may be ω-nonstandard, then you need a single parameter, for the length of the sequence. If all worlds are ω-standard, then this holds without admitting parameters.)

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S4 is always a lower bound. To get that it also an upper bound uses that the class of finite pre-trees is complete for S4. That is, if a modal assertion φ is not in S4 then there is a finite pre-tree which invalidates φ.

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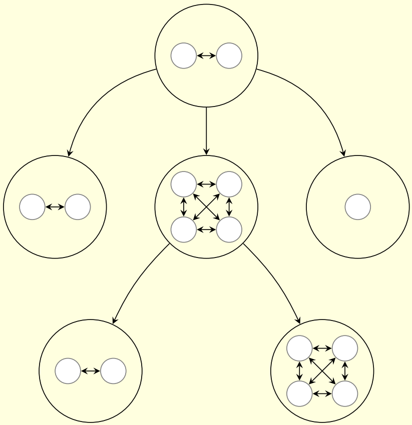
If a potentialist system admits a universal finite sequence, then its modal validities are precisely S4. So it has branching modalities.
(If worlds may be ω-nonstandard, then you need a single parameter, for the length of the sequence. If all worlds are ω-standard, then this holds without admitting parameters.)

S4 is always a lower bound. To get that it also an upper bound uses that the class of finite pre-trees is complete for S4. That is, if a modal assertion φ is not in S4 then there is a finite pre-tree which invalidates φ.

So to prove the theorem we have to see how to use a universal finite sequence to label pre-trees with formulae so that the order-relation on the pre-tree agrees with possibility among the formulae.

The universal finite sequence and labeling pre-trees

Look at what gets added to the end of the universal finite sequence:



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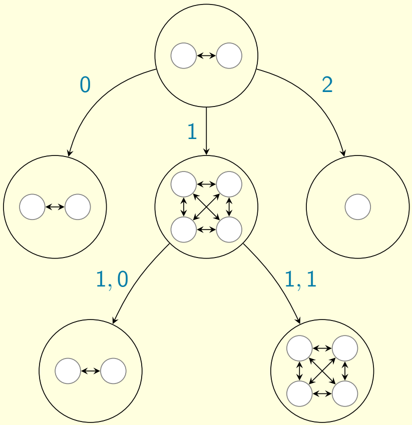

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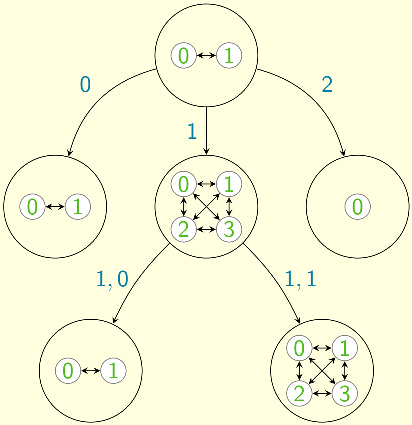
- Step 2: the final odd number o on the sequence tells you where in your cluster you are. If K is the size of the cluster, then $o - 1 \bmod 2K$ identifies your node in the cluster. (If no odd numbers are on the sequence, default to 0.)



Characterizing end-extensional possibility

Theorem (Hamkins–Welch–W.)

The following are equivalent, for countable ω-nonstandard $M \models \mathsf{ZF}$, and φ a sentence.

1. $M \models \Diamond \varphi$ *in end-extensional potentialism.*
2. *For each standard k, M thinks that each countable transitive has an end-extension to a model of $\mathsf{ZF}_k + \varphi$.*
3. *For each standard k, M thinks that each real is in an ω-model of $\mathsf{ZF}_k + \varphi$.*
4. φ *is consistent with* ZF *plus the* Σ_1*-theory of* M.

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Question

Does every countable ω*-nonstandard model of* ZF *have an end-extension which satisfies the end-extensional maximality principle?*

Thank you!